XXI. *On Skew Surfaces, otherwise Scrolls.* *By* ARTHUR CAYLEY, *F.R.S.*

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IT may be convenient to mention at the outset that, in the paper "On the Theory of Skew Surfaces"*, I pointed out that upon any skew surface of the order n there is a singular (or nodal) curve meeting each generating line in $(n-2)$ points, and that the class of the circumscribed cone (or, what is the same thing, the class of the surface) is equal to the order n of the surface. In the paper "On a Class of Ruled Surfaces"†, Dr. SALMON considered the surface generated by a line which meets three curves of the orders m, n, p respectively: such surface is there shown to be of the order $=2mnp$; and it is noticed that there are upon it a certain number of double right lines (nodal generators); to determine the number of these, it was necessary to consider the skew surface generated by a line meeting a given right line and a given curve of the order m twice; and the order of such surface is found to be $=\frac{1}{2}m(m-1)+h$, where h is the number of apparent double points of the curve. The theory is somewhat further developed in Dr. SALMON'S memoir "On the Degree of a Surface reciprocal to a given one"‡, where certain minor limits are given for the orders of the nodal curves on the skew surface generated by a line meeting a given right line and two curves of the orders m and n respectively, and on that generated by a line meeting a given right line and a curve of the order m twice. And in the same memoir the author considers the skew surface generated by a line the equations whereof are $(a, \ldots \between t, 1)^m=0$ $(a', \ldots \between t, 1)^n=0$, where $a, \ldots a', \ldots$ are any linear functions of the coordinates, and t is an arbitrary parameter. And the same theories are reproduced in the 'Treatise on the Analytic Geometry of Three Dimensions'§. I will also, though it is less closely connected with the subject of the present memoir, refer to a paper by M. CHASLES, "Description des Courbes à double courbure de tous les ordres sur les surfaces réglées du troisième et du quatrième ordre"‖.

The present memoir (in the composition of which I have been assisted by a correspondence with Dr. SALMON) contains a further development of the theory of the skew surfaces generated by a line which meets a given curve or curves: viz. I consider, 1st, the surface generated by a line which meets each of three given curves of the orders m, n, p respectively; 2nd, the surface generated by a line which meets a given curve of the order m twice, and a given curve of the order n once; 3rd, the surface which meets

* Cambridge and Dublin Math. Journ. vol. vi. pp. 171–173 (1852).

† Ibid. vol. viii. pp. 45, 46 (1853).

‡ Trans. Royal Irish Acad. vol. xxiii. pp. 461–488 (read 1855).

§ Dublin, 1862.

‖ Comptes Rendus, t. liii. (1861, 2e Sem.), pp. 884–889.

a given curve of the order m three times; or, as it is very convenient to express it, I consider the skew surfaces, or say the "Scrolls," $S(m, n, p)$, $S(m^2, n)$, $S(m^3)$. The chief results are embodied in the Table given after this introduction, at the commencement of the memoir. It is to be noticed that I attend throughout to the general theory, not considering otherwise than incidentally the effect of any singularity in the system of the given curves, or in the given curves separately: the memoir contains however some remarks as to what are the singularities material to a complete theory; and, in particular as regards the surface $S(m^3)$, I am thus led to mention an entirely new kind of singularity of a curve in space—viz. such a curve has in general a determinate number of "lines through four points" (lines which meet the curve in four points); it may happen that, of the lines through three points which can be drawn through any point whatever of the curve, a certain number will unite together and form a line through four (or more) points, the number of the lines through four points (or through a greater number of points) so becoming infinite.

Notation and Table of Results, Articles 1 to 10.

1. In the present memoir a letter such as m denotes the order of a curve in space. It is for the most part assumed that the curve has no actual double points or stationary points, and the corresponding letter M denotes the class of the curve taken negatively and divided by 2; that is, if h be the number of apparent double points, then $M=-\frac{1}{2}[m]^2+h$: here and elsewhere $[m]^2$, &c. denote factorials, viz. $[m]^2=m(m-1)$, $[m]^3=m(m-1)(m-2)$, &c. It is to be noticed that for the system of two curves m, m', if h, h' represent the number of apparent double points of the two curves respectively, then for the system the number of apparent double points is $=mm'+h+h'$, and the corresponding value of M is therefore $-\frac{1}{2}[m+m']^2+mm'+h+h'$, which is $=-\frac{1}{2}[m]^2+h-\frac{1}{2}[m']^2+h'$, which is $=M+M'$.

2. The use of the combinations (m, n, p, q), (m^2, n, p), &c. hardly requires explanation; it may however be noticed that $G(m, n, p, q)$ denoting the lines which meet the curves m, n, p, q (that is, curves of these orders) each of them once, $G(m^2, n, p)$ will denote the lines which meet the curve m twice and the curves n and p each of them once; and so in all similar cases.

3. The letters G, S, ND, NG, NR, NT (read Generators, Scroll, Nodal Director, Nodal Generator, Nodal Residue, and Nodal Total) are in the nature of functional symbols, used (according to the context) to denote geometrical forms, or else the orders of these forms. Thus $G(m, n, p, q)$ denotes either the lines meeting the curves m, n, p, q each of them once, or else it denotes the order of such system of lines, that is, the number of lines. And so $S(m, n, p)$ denotes the Skew Surface or Scroll generated by a line which meets the curves m, n, p each once, or else it denotes the order of such surface.

4. $G(m, n, p, q)$: the signification is explained above.

5. $S(m, n, p)$: the signification has just been explained; but as the surfaces $S(m, n, p)$,

$S(m^2, n)$, $S(m^3)$ are in fact the subject of the present memoir, I give the explanation in full for each of them, viz. $S(m, n, p)$ is the surface generated by a line which meets the curves m, n, p each once; $S(m^2, n)$ is the surface generated by a line which meets the curve m twice and the curve n once; $S(m^3)$ the surface generated by the line which meets the curve m thrice. As already mentioned, these surfaces and their orders are represented by the same symbols respectively.

6. $ND(m, n, p)$. The directrix curves m, n, p of the scroll $S(m, n, p)$ are nodal (multiple) curves on the surface, viz. m is an np-tuple curve, and so for n and p. Reckoning each curve according to its multiplicity, viz. the curve m being reckoned $\frac{1}{2}[np]^2$ times, or as of the order $m.\frac{1}{2}[np]^2$, and so for the curves n and p, the aggregate, or sum of the orders, gives the Nodal Director $ND(m, n, p)$.

7. $NG(m, n, p)$. The scroll $S(m, n, p)$ has the nodal generating lines $G(m^2, n, p)$, $G(m, n^2, p)$, $G(m, n, p^2)$. Each of these is a mere double line, to be reckoned once only, and we have thus the Nodal Generator

$$NG(m, n, p)=G(m^2, n, p)+G(m, n^2, p)+G(m, n, p^2).$$

But to take another example, the scroll $S(m^2, n)$ has the nodal generating lines $G(m^3, n)$, each of which is a triple line to be reckoned $\frac{1}{2}[3]^2$, that is, three times, and also the nodal generating lines $G(m^2, n^2)$, each of them a mere double line to be reckoned once only; whence here $NG(m^2, n)=3G(m^3, n)+G(m^2, n^2)$. And so for the scroll $S(m^3)$, this has the nodal generating lines $G(m^4)$, each of them a quadruple line to be reckoned $\frac{1}{2}[4]^2$, that is, six times; or we have $NG(m^3)=6G(m^4)$.

8. $NR(m, n, p)$. The scroll $S(m, n, p)$ has besides the directrix curves m, n, p or Nodal Director, and the nodal generating lines or Nodal Generator, a remaining nodal curve or Nodal Residue, the locus of the intersections of two non-coincident generating lines meeting in a point not situate on any one of the directrix curves. This Nodal Residue, as well for the scroll $S(m, n, p)$ as for the scrolls $S(m^2, n)$ and $S(m^3)$ respectively, is a mere double curve to be reckoned once only; and such curve or its order is denoted by NR, viz. for the scroll $S(m, n, p)$, the Nodal Residue is $NR(m, n, p)$.

9. $NT(m, n, p)$. The Nodal Director, Nodal Generator, and Nodal Residue of the scroll $S(m, n, p)$ form together the Nodal Total $NT(m, n, p)$, that is, we have

$$NT(m, n, p)=ND(m, n, p)+NG(m, n, p)+NR(m, n, p);$$

and similarly for the scrolls $S(m^2, n)$ and $S(m^3)$.

10. I remark that the formulæ are best exhibited in an order different from that in which they are in the sequel obtained, viz. I collect them in the following

Table.

$$\begin{aligned}
G(m, n, p, q)&=2mnpq,\\
G(m^2, n, p)&=np([m]^2+M),\\
G(m^2, n^2)&=\tfrac{1}{2}[m]^2[n]^2+M.\tfrac{1}{2}[n]^2+N.\tfrac{1}{2}[m]^2+MN,\\
G(m^3, n)&=n\left(\tfrac{1}{3}[m]^3+M(m-2)\right),\\
G(m^4)&=\tfrac{1}{12}[m]^4+m+M(\tfrac{1}{2}[m]^2-2m+\tfrac{11}{2})+M^2.\tfrac{1}{2},
\end{aligned}$$

$$
\begin{aligned}
\mathrm{S}(m, n, p) &= 2mnp, \\
\mathrm{ND}(m, n, p) &= \tfrac{1}{2}mnp(mn+mp+np-3), \\
\mathrm{NG}(m, n, p) &= mnp(m+n+p-3)+\mathrm{M}np+\mathrm{N}mp+\mathrm{P}mn, \\
\mathrm{NR}(m, n, p) &= \tfrac{1}{2}mnp\big(4mnp-(mn+mp+np)-2(m+n+p)+5\big), \\
{}^*\mathrm{NT}(m, n, p) &= \tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+\mathrm{M}np+\mathrm{N}mp+\mathrm{P}mn \\
&= 2mnp(mnp-1)+\mathrm{M}np+\mathrm{N}mp+\mathrm{P}mn;
\end{aligned}
$$

included in which we have

$$
\begin{aligned}
\mathrm{S}(1, 1, m) &= 2m, \\
\mathrm{ND}(1, 1, m) &= [m]^2, \\
\mathrm{NG}(1, 1, m) &= [m]^2+\mathrm{M}, \\
\mathrm{NR}(1, 1, m) &= 0, \\
\mathrm{NT}(1, 1, m) &= \tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+\mathrm{M} \\
&= 2[m]^2+\mathrm{M},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{S}(1, m, n) &= 2mn, \\
\mathrm{ND}(1, m, n) &= \tfrac{1}{2}mn(mn+m+n-3), \\
\mathrm{NG}(1, m, n) &= mn(m+n-2)+\mathrm{M}n+\mathrm{N}m, \\
\mathrm{NR}(1, m, n) &= \tfrac{3}{2}[m]^2[n]^2, \\
\mathrm{NT}(1, m, n) &= \tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+\mathrm{M}n+\mathrm{N}m \\
&= 2[mn]^2+\mathrm{M}n+\mathrm{N}m.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
\mathrm{S}(m^2, n) &= n\,\big([m]^2+\mathrm{M}\big), \\
\mathrm{ND}(m^2, n) &= n\,\big(\tfrac{1}{8}[m]^4+[m]^3+\mathrm{M}(\tfrac{1}{2}[m]^2-\tfrac{1}{2})+\mathrm{M}^2.\tfrac{1}{2}\big) \\
&\quad +[n]^2(\tfrac{1}{2}[m]^3+\tfrac{1}{2}[m]^2), \\
\mathrm{NG}(m^2, n) &= n\,\big([m]^3+\mathrm{M}.3(m-2)\big) \\
&\quad +[n]^2(\tfrac{1}{2}[m]^2+\tfrac{1}{2}\mathrm{M}) \\
&\quad +\mathrm{N}\,(\tfrac{1}{2}[m]^2+\mathrm{M}), \\
\mathrm{NR}(m^2, n) &= n\,\big(\tfrac{3}{8}[m]^4+\mathrm{M}(\tfrac{1}{2}[m]^2-2m+3)\big) \\
&\quad +[n]^2\big(\tfrac{1}{2}[m]^4+\tfrac{3}{2}[m]^3+[m]^2+\mathrm{M}([m]^2-\tfrac{1}{2})+\mathrm{M}^2.\tfrac{1}{2}\big), \\
\mathrm{NT}(m^2, n) &= \tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+n\mathrm{M}(m-\tfrac{5}{2})+\mathrm{N}(\tfrac{1}{2}[m]^2+\mathrm{M}) \\
&= n\,\big(\tfrac{1}{2}[m]^4+2[m]^3+\mathrm{M}([m]^2+m-\tfrac{7}{2})+\mathrm{M}^2.\tfrac{1}{2}\big) \\
&\quad +[n]^2(\tfrac{1}{2}[m]^4+2[m]^3+[m]^2+\mathrm{M}.[m]^2+\mathrm{M}^2.\tfrac{1}{2}) \\
&\quad +\mathrm{N}\,(\tfrac{1}{2}[m]^2+\mathrm{M});
\end{aligned}
$$

* In the first of the two expressions for $\mathrm{NT}(m, n, p)$, S stands for $\mathrm{S}(m, n, p)$; and so in the first of the two expressions for $\mathrm{NT}(m^2, n)$, &c., S stands for $\mathrm{S}(m^2, n)$, &c.

included in which we have

$$S(1, m^2)=[m]^2+M,$$
$$ND(1, m^2)=\tfrac{1}{8}[m]^4+[m]^3+M(\tfrac{1}{2}[m]^2-\tfrac{1}{2})+M^2.\tfrac{1}{2},$$
$$NG(1, m^2)=[m]^3+M.3(m-2),$$
$$NR(1, m^2)=\tfrac{3}{8}[m]^4+M(\tfrac{1}{2}[m]^2-2m+3),$$
$$NT(1, m^2)=\tfrac{1}{2}S^2-S+M(m-\tfrac{5}{2})$$
$$=\tfrac{1}{2}[m]^4+2[m]^3+M([m]^2+m-\tfrac{7}{2})+M^2.\tfrac{1}{2};$$

and finally

$$S(m^3)=\tfrac{1}{3}[m]^3+(m-2)M,$$
$$ND(m^3)=\tfrac{1}{8}[m]^5+\tfrac{1}{2}[m]^4+\tfrac{1}{2}[m]^3+M(\tfrac{1}{2}[m]^3+\tfrac{1}{2}[m])+M^2.\tfrac{1}{2}m,$$
$$NG(m^3)=\tfrac{1}{2}[m]^4+6m+M(3[m]^2-12m+33)+M^2.3,$$
$$NR(m^3)=\tfrac{1}{18}[m]^6+\tfrac{3}{8}[m]^5-\tfrac{1}{2}[m]^3+3m$$
$$+M(\tfrac{1}{3}[m]^4-\tfrac{1}{6}[m]^3-\tfrac{5}{2}[m]^2+8m-20)+M^2(\tfrac{1}{2}[m]^2-2m),$$
$$NT(m^3)=\tfrac{1}{2}S^2-S+3m+M(\tfrac{1}{2}[m]^2-\tfrac{5}{2}m+11)+M^2$$
$$=\tfrac{1}{18}[m]^6+\tfrac{1}{2}[m]^5+[m]^4+3m$$
$$+M(\tfrac{1}{3}[m]^4+\tfrac{1}{3}[m]^3+\tfrac{1}{2}[m]^2-\tfrac{7}{2}m+13)$$
$$+M^2(\tfrac{1}{2}[m]^2-\tfrac{3}{2}m+3).$$

The formulæ are investigated in the following order, ND, G, NG, S, NR, and NT.

The ND *formulæ*, Articles 11 to 13.

11. $ND(m, n, p)$.—Taking any point on the curve m, this is the vertex of two cones passing through the curves n, p respectively; the cones are of the orders n, p respectively, and they intersect therefore in np lines, which are the generating lines through the point on the curve m; hence this curve is an np-tuple line on the scroll $S(m, n, p)$, and we have thus the term $m.\tfrac{1}{2}[np]^2$ of ND. Whence

$$ND(m, n, p)=m.\tfrac{1}{2}[np]^2+n.\tfrac{1}{2}[mp]^2+p.\tfrac{1}{2}[mn]^2$$
$$=\tfrac{1}{2}mnp(mn+mp+np-3).$$

12. $ND(m^2, n)$.—Taking first a point on the curve m, this is the vertex of a cone of the order $m-1$ through the curve m, and of a cone of the order n through the curve n; the two cones intersect in $(m-1)n$ lines, which are the generating lines through the point on the curve m; that is, the curve m is a $(m-1)n$-tuple line on the scroll $S(m^2, n)$; and we have thus the term $m.\tfrac{1}{2}[(m-1)n]^2$ of ND. Taking next a point on the curve n, this is the vertex of a cone of the order m through the curve m; such cone has $(h=)\tfrac{1}{2}[m]^2+M$ double lines, which are the generating lines through the point on the curve n; hence this curve is a $(\tfrac{1}{2}[m]^2+M)$tuple line on the surface, and we have thus the term $n.\tfrac{1}{2}\left[\tfrac{1}{2}[m]^2+M\right]^2$ in ND. And therefore

$$ND(m^2, n)=m.\tfrac{1}{2}[(m-1)n]^2+n.\tfrac{1}{2}\left[\tfrac{1}{2}[m]^2+M\right]^2$$
$$=n.\left(\tfrac{1}{8}[m]^4+[m]^3+M(\tfrac{1}{2}[m]^2-\tfrac{1}{2})+M^2.\tfrac{1}{2}\right)$$
$$+[n]^2(\tfrac{1}{2}[m]^3+\tfrac{1}{2}[m]^2).$$

13. $\mathrm{ND}(m^3)$.—Taking a point on the curve m, this is the vertex of a cone of the order $m-1$ through the curve m; such cone has $(h-m+2=)\frac{1}{2}[m]^2-m+2+\mathrm{M}$ double lines, or the curve m is a $(\frac{1}{2}[m]^2-m+2+\mathrm{M})$tuple line on the scroll $\mathrm{S}(m)^3)$. Hence we have

$$\mathrm{ND}(m)^3=m.\tfrac{1}{2}\left[\tfrac{1}{2}[m]^2-m+2+\mathrm{M}\right]^2$$
$$=\tfrac{1}{8}[m]^5+\tfrac{1}{2}[m]^4+\tfrac{1}{2}[m]^3+\mathrm{M}(\tfrac{1}{2}[m]^3+\tfrac{1}{2}[m])+\mathrm{M}^2.\tfrac{1}{2}m.$$

Preparatory remarks in regard to the G *formulæ, the hypertriadic singularities of a curve in space*, Articles 14 to 22.

14. It is to be remarked that the generating line of any one of the scrolls $\mathrm{S}(m, n, p)$, $\mathrm{S}(m^2, n)$, $\mathrm{S}(m^3)$ satisfies three conditions; and that it cannot in anywise happen that one of these conditions is implied in the other two. Thus, for instance, as regards the scroll $\mathrm{S}(m, n, p)$, if the curves m, n are given, and we take the entire series of lines meeting each of these curves, these lines form a double series of lines, all of them passing of course through the curves m, n, but not all of them passing through any other curve whatever; that is, there is no curve p such that every line passing through the curves m and n passes also through the curve p. And the like as regards the scrolls $\mathrm{S}(m^2, n)$ and $\mathrm{S}(m^3)$.

15. But (in contrast to this) if the three conditions are satisfied, it may very well happen that a fourth condition is satisfied *ipso facto*. To see how this is, imagine a curve q on the scroll $\mathrm{S}(m, n, p)$, or, to meet an objection which might be raised, say a curve q the complete intersection of the scroll $\mathrm{S}(m, n, p)$ by a plane or any other surface. Every line whatever which meets the curves m, n, p is a generating line of the scroll $\mathrm{S}(m, n, p)$, and as such will meet the curve q; that is, in the case in question, $\mathrm{G}(m, n, p, q)$, the lines which meet the curves m, n, p, q, are the entire series of generating lines of the scroll $\mathrm{S}(m, n, p)$, and they are thus infinite in number; so that in such case the question does not arise of finding the number of the lines $\mathrm{G}(m, n, p, q)$. The like remarks apply to the lines $\mathrm{G}(m^2, n, p)$, $\mathrm{G}(m^2, n^2)$, $\mathrm{G}(m^3, n)$, and $\mathrm{G}(m^4)$; but I will develope them somewhat more particularly as regards the lines $\mathrm{G}(m^4)$.

16. Given a curve m, then (as in fact mentioned in the investigation for $\mathrm{ND}(m^3)$) through *any point whatever* of the curve there can be drawn

$$(h-m+2=)[m]^2+m-2+\mathrm{M}$$

lines meeting the curve in two other points, or say $[m]^2+m-2+\mathrm{M}$ lines through three points. But in general no one of these lines meets the curve in a fourth point; that is, we cannot through every point of the curve m draw a line through four points; there are, however, on the curve m a *certain* number $(=4\mathrm{G}(m^4))$ of points through which can be drawn a line through four points, or line $\mathrm{G}(m^4)$.

17. But the curve m may be such that through every point of the curve there passes a line through four points. In fact, assume any skew surface or scroll whatever, and upon this surface a curve meeting each generating line in four points (*e. g.* the intersection of the scroll by a quartic surface). Taking the curve in question for the curve m, then it is clear that through every point of this curve there passes a line (the generating line of the assumed scroll) which is a line through four points, or line $\mathrm{G}(m^4)$.

18. It is to be noticed, moreover, that if we take on the curve m any point whatever, then of the $[m]^2+m-2+\mathrm{M}$ lines through three points which can be drawn through this point, three will unite together in the generating line of the assumed scroll (for if 0 be the point on the curve m, and 1, 2, 3 the other points in which the generating line of the assumed scroll meets the curve m, then such generating line unites the three lines 012, 013, 023, each of them a line through three points); and there will be besides $\frac{1}{2}[m]^2+m-5+\mathrm{M}$ mere lines through three points. The line through four points generates the assumed scroll taken $(\frac{1}{2}[3]^2=)$ 3 times, or considered as three coincident scrolls; the remaining lines generate a scroll $\mathrm{S}'(m^3)$, which is such that the curve m is on this scroll a $(\frac{1}{2}[m]^2+m-5+\mathrm{M})$tuple line; the assumed scroll three times and the scroll $\mathrm{S}'(m^3)$ make up the entire scroll $\mathrm{S}(m^3)$ derived from the curve m, or say $\mathrm{S}(m^3)=$ 3 (assumed scroll) $+\mathrm{S}'(m^3)$.

19. The case just considered is that of a curve m such that through every point of it there passes a line through four points counting as $(\frac{1}{2}[3]^2=)$ 3 lines through three points, and that all the other lines through three points are mere lines through three points. But it is clear that we may in like manner have a line through p points counting as $\frac{1}{2}[p-1]^2$ lines through three points; and more generally if p, q, ... are numbers all different and not <3, and if

$$\tfrac{1}{2}[m]^2-m+2+\mathrm{M}=\alpha\,.\,\tfrac{1}{2}[p-1]^2+\beta\,.\,\tfrac{1}{2}[q-1]^2+\ldots,$$

then we may have a curve m such that through every point of it there pass α lines each through p points and counting as $\frac{1}{2}[p-1]^2$ lines, β lines each through q points and counting as $\frac{1}{2}[q-1]^2$ lines, &c....: the case $p=3$ gives of course α lines each through three points and counting as a single line. It is to be added that, in the case just referred to, the α lines will generate a scroll $\mathrm{S}'(m^3)$ taken $\frac{1}{6}[p]^3$ times, the β lines will generate a scroll $\mathrm{S}''(m^3)$ taken $\frac{1}{6}[q]^3$ times, &c., which scrolls together make up the scroll $\mathrm{S}(m^3)$, or say

$$\mathrm{S}(m^3)=\tfrac{1}{6}[p]^3\,.\,\mathrm{S}'(m^3)+\tfrac{1}{6}[q]^3\,.\,\mathrm{S}''(m^3)+\ \&\text{c.};$$

it may however happen that, *e. g.* of the α lines, any set or sets or even each line will generate a distinct scroll or scrolls—that is, that the scroll $\mathrm{S}'(m^3)$ will itself break up into scrolls of inferior orders.

20. A good illustration is afforded by taking for the curve m a curve on the hyperboloid or quadric scroll*; such curves divide themselves into species; viz. we have say the (p, q) curve on the hyperboloid, a curve of the order $p+q$ meeting each generating line of the one kind in p points, and each generating line of the other kind in q points; here

$$m=p+q,\ (h=\tfrac{1}{2}[p]^2+\tfrac{1}{2}[q]^2,\text{ and }\therefore)\mathrm{M}=-pq.$$

Assuming for the moment that p, q are each of them not less than 3, it is clear that the lines through three points which can be drawn through any point of the curve are the generating line of the one kind counting as $\frac{1}{2}[p-1]^2$ lines through three points,

* It is hardly necessary to remark that (*reality* being disregarded) any quadric surface whatever is a hyperboloid or quadric scroll.

and the generating line of the other kind counting as $\frac{1}{2}[q-1]^2$ lines through three points, so that

$$\tfrac{1}{2}[m]^2+m-2+\mathrm{M}=\tfrac{1}{2}[p-1]^2+\tfrac{1}{2}[q-1]^2.$$

The complete scroll $\mathrm{S}(m^3)$ is made up of the hyperboloid considered as generated by the generating lines of the one kind taken $\frac{1}{6}[p]^3$ times, and the hyperboloid considered as generated by the generating lines of the other kind taken $\frac{1}{6}[q]^3$ times (so that there is in this case the speciality that the surfaces $\mathrm{S}'(m^3)$, $\mathrm{S}''(m^3)$ are in fact the same surface). And hence we have

$$\mathrm{S}(m^3)=\big(2(\tfrac{1}{6}[p]^3+\tfrac{1}{6}[q]^3)=\big)\tfrac{1}{3}[p]^3+\tfrac{1}{3}[q]^3.$$

21. I notice also the case of a system of m lines. Taking here a point on one of the lines, the $(h-m+2=)[m]^2-m+2$ lines through three points which can be drawn through this point are the $\frac{1}{2}[m-1]^2$ lines which can be drawn meeting a pair of the other $(m-1)$ lines, and besides this the line itself counting as one line through three points $(\frac{1}{2}[m-1]^2+1=[m]^2-m+2)$; the line itself, thus counting as a single line through three points, is not to be reckoned as a line through four or more points drawn through the point in question, that is, the system is not to be regarded as a curve through every point of which there passes a line through four points: each of the lines is nevertheless to be counted as a single line through four points, and (since there are besides two lines which may be drawn meeting each four of the m lines) the total number of lines through four points is $=\frac{1}{12}[m]^4+m$.

22. In the following investigations for $\mathrm{G}(m, n, p, q)$, &c., the foregoing special cases are excluded from consideration; it may however be right to notice how it is that the formulæ obtained are inapplicable to these special cases; for instance, as will immediately be seen, the number of the lines $\mathrm{G}(m, n, p, q)$ is obtained as the number of intersections of the surface $\mathrm{S}(m, n, p)$ by the curve q, $=2mnp\times q=2mnpq$; but if the curve q lie on the surface $\mathrm{S}(m, n, p)$, then $\mathrm{G}(m, n, p, q)$ is no longer $=2mnpq$.

The G *formulæ*, Articles 23 to 34.

23. $\mathrm{G}(m, n, p, q)$.—Considering the scroll $\mathrm{S}(m, n, p)$ generated by a line which meets each of the curves m, n, p, this meets the curve q in $q\ \mathrm{S}(m, n, p)$ points through each of which there passes a line $\mathrm{G}(m, n, p, q)$; that is, we have

$$\mathrm{G}(m, n, p, q)=q\ \mathrm{S}(m, n, p).$$

But from this equation we have

$$\mathrm{S}(m, n, p)=\mathrm{G}(1, m, n, p)=p\ \mathrm{S}(1, m, n);$$

thence also

$$\mathrm{S}(1, m, n)=\mathrm{G}(1, 1, m, n)=n\ \mathrm{S}(1, 1, n),$$

and

$$\mathrm{S}(1, 1, m)=\mathrm{G}(1, 1, 1, m)=m\ \mathrm{S}(1, 1, 1);\ \ \mathrm{S}(1, 1, 1)=\mathrm{G}(1, 1, 1, 1)=2,$$

since 2 is the number of lines which can be drawn meeting each of four given right lines. Hence ultimately

$$G(m, n, p, q)=mnpqG(1, 1, 1, 1)=2mnpq.$$

24. $G(m^2, n, p)$.—In a precisely similar manner we find

$$G(m^2, n, p)=npG(1, 1, m^2)=npS(1, m^2),$$

and it is the same question to find $G(1, 1, m^2)$ and to find $S(1, m^2)$. I investigate $G(1, 1, m^2)$ by considering the particular case where the curve m is a plane curve having n double points. The plane of the curve meets the two lines 1, 1 in two points, and the line through these two points meets each of the lines 1, 1, and meets the curve in m points; combining the last-mentioned m points two and two together, the line in question is to be considered as $\frac{1}{2}[m]^2$ coincident lines, each of them meeting the lines 1, 1, and also meeting the curve m twice. But we may also through any double point of the curve draw a line meeting each of the lines 1, 1; such line, inasmuch as it passes through a double point, meets the curve twice; and we have h such lines. This gives for the case in question $G(1, 1, m^2)=h+\frac{1}{2}[m]^2$; or, introducing in the place of h the quantity $M(=h-\frac{1}{2}[m]^2)$, so that $h=\frac{1}{2}[m]^2+M$, we have

$$G(1, 1, m^2)=[m]^2+M.$$

And, to the double points of the plane curve, there correspond in the general case the apparent double points of the curve m. Admitting the correctness of the result just obtained, we then have

$$G(m^2, n, p)=np([m]^2+M).$$

25. $G(m^2, n^2)$.—I investigate the value by a process similar to that employed for $G(1, 1, m^2)$. Suppose that the curves m and n are plane curves having respectively h and k double points; then the line of intersection of the two planes meets the curve m in m points, and the curve n in n points; or, combining in every manner the m points two and two together, and the n points two and two together, the line in question is to be considered as $\frac{1}{2}[m]^2 . \frac{1}{2}[n]^2$ coincident lines, each meeting the curves m, n, each curve twice. There are besides the hk lines joining each double point of the curve m with each double point of the curve n. This gives in all $\frac{1}{4}[m]^2[n]^2+hk$ lines; or, writing $h=\frac{1}{2}[m]^2+M$, $k=\frac{1}{2}[n]^2+N$, the number is

$$=\tfrac{1}{2}[m]^2[n]^2+M . \tfrac{1}{2}[n]^2+N . \tfrac{1}{2}[m]^2+MN;$$

which is the value of $G(m^2, n^2)$ given by the investigation.

26. $G(m^3, n)$.—We have

$$G(m^3, n)=nG(1, m^3)=nS(m^3),$$

and it is in fact the same question to find $G(1, m^3)$ and to find $S(m^3)$. I assume for the present that the value is $=\frac{1}{3}[m]^3+M(m-2)$; and we then have

$$G(m^3, n)=n\left(\tfrac{1}{3}[m]^3+M(m-2)\right),$$

27. Before going further, I observe that there are certain functional conditions which must be satisfied by the G formulæ. Thus if the curve m be replaced by the system of the two curves m, m', instead of M we have $\mathrm{M}+\mathrm{M}'$. Let $\mathrm{G}(m)$ denote any one of the functions $\mathrm{G}(m, n, p, q)$, $\mathrm{G}(m, n^2, p)$, $\mathrm{G}(m, n^3)$, we must have

$$\mathrm{G}(m+m')=\mathrm{G}(m)+\mathrm{G}(m').$$

Similarly, if $\mathrm{G}(m^2)$ denote either of the functions $\mathrm{G}(m^2, n, p)$, $\mathrm{G}(m^2, n^2)$, we must have

$$\mathrm{G}(m+m')^2=\mathrm{G}(m^2)+\mathrm{G}(m, m')+\mathrm{G}(m'^2);$$

and so if $\mathrm{G}(m^3)$ stand for $\mathrm{G}(m^3, n)$, then

$$\mathrm{G}(m+m')^3=\mathrm{G}(m^3)+\mathrm{G}(m^2, m')+\mathrm{G}(m, m'^2)+\mathrm{G}(m'^3);$$

and finally

$$\mathrm{G}(m+m')^4=\mathrm{G}(m^4)+\mathrm{G}(m^3, m')+\mathrm{G}(m^2, m'^2)+\mathrm{G}(m, m'^3)+\mathrm{G}(m'^4).$$

28. The first three equations may be at once verified by means of the above given values of the G functions. But conversely, at least on the assumption that $\mathrm{G}(m)$, $\mathrm{G}(m^2)$, &c., in so far as they respectively depend on the curve m, are functions of m and M only, we may, by the solution of the functional equations, obtain the values of the G functions. It is to be observed that the first equation is of the form

$$\phi(m+m')=\phi(m)+\phi(m'),$$

the general solution whereof is

$$\phi m=\alpha m+\beta\mathrm{M};$$

the second equation, supposing that $\mathrm{G}\,(m, m')$ is known—the third equation, supposing that $\mathrm{G}(m^2, m')$ and $\mathrm{G}(m, m'^2)$ are known—and the fourth equation, supposing that $\mathrm{G}(m^3, m')$, $\mathrm{G}(m^2, m'^2)$, $\mathrm{G}(m, m'^3)$ are known, are respectively of the form

$$\phi(m+m')=\phi m+\phi m'+\text{funct.}\,(m, m');$$

and hence if a particular solution be given, the general solution is

$$\phi(m)=\text{Particular Solution}+\alpha m+\beta\mathrm{M}.$$

The values of the constants must in each case be determined by special considerations.

29. The value of $\mathrm{G}(m, n, p, q)$ was obtained strictly; that of $\mathrm{G}(m^2, n, p)$ was reduced to depend on $\mathrm{G}(1, 1, m^2)$, and that of $\mathrm{G}(m^3, n)$ on $\mathrm{G}(1, m^3)$. I apply therefore the functional equations to the confirmation of the values of $\mathrm{G}(1, 1, m^2)$, $\mathrm{G}(m^2, n^2)$, and $\mathrm{G}(1, m^3)$, and to the determination of the value of $\mathrm{G}(m^4)$.

30. First, if $\mathrm{G}(m^2)$ denote $\mathrm{G}(1, 1, m^2)$, then $\mathrm{G}(m, m')$ denotes $\mathrm{G}(1, 1, m, m')$, which is $=2mm'$; hence

$$\mathrm{G}(m+m')^2-\mathrm{G}(m^2)-\mathrm{G}(m'^2)=2mm',$$

which is satisfied by $\mathrm{G}(m^2)=[m]^2$. This gives

$$\mathrm{G}(1, 1, m^2)=[m]^2+\alpha m+\beta\mathrm{M}.$$

But if the curve m be a system of m lines ($m=m$, $\mathrm{M}=0$), then $\mathrm{G}(1, 1, m^2)=[m]^2$; and

again, if the curve m be a conic ($m=2$, $M=-1$), then $G(1, 1, m^2)=1$. This gives $\alpha=0$, $\beta=1$, and therefore

$$G(1, 1, m^2)=[m]^2+M.$$

31. Next, if $G(m^2)$ denote $G(m^2, n^2)$, then $G(m, m')$ denotes $G(m, m', n^2)$, which is $=mm'([n]^2+N)$. The functional equation is

$$G(m+m')^2-G(m^2)-G(m'^2)=mm'([n]^2+N),$$

which is satisfied by

$$G(m^2)=\tfrac{1}{2}[m]^2([n]^2+N).$$

Hence we have

$$G(m^2, n^2)=\tfrac{1}{2}[m]^2([n]^2+N)+\alpha m+\beta M,$$

where α, β are functions of n, N; and observing that $G(m^2, n^2)$ must be symmetrical in regard to the curves m and n, it is easy to see that we may write

$$G(m^2, n^2)=\tfrac{1}{2}[m]^2[n]^2+M\,.\,\tfrac{1}{2}[n]^2+N\,.\,\tfrac{1}{2}[m]^2+\alpha mn+\beta(mN+nM)+\gamma MN,$$

where α, β, γ are absolute constants. To determine them, if the curve m be a pair of lines ($m=2$, $M=0$), then

$$G(m^2, n^2)=G(1, 1, n^2)=[n]^2+N;$$

and if each of the curves m, n be a conic ($m=2$, $M=-1$, $n=2$, $N=-1$), then

$$G(m^2, n^2)=1.$$

These cases give $\alpha=\beta=0$, $\gamma=1$, and therefore

$$G(m^2, n^2)=\tfrac{1}{2}[m]^2[n]^2+M\,.\,\tfrac{1}{2}[n]^2+N\,.\,\tfrac{1}{2}[m]^2+MN.$$

32. Again, $G(m^3)$ standing for $G(1, m^3)$, then $G(m^2, m')$ and $G(m, m'^2)$ will stand for $G(1, m^2, m')$ and $G(1, m, m'^2)$, the values whereof are $m'([m]^2+M)$ and $m([m']^2+M')$ respectively. We thus have

$$G(m+m')^3-G(m^3)-G(m'^3)=m'([m]^2+M)+m([m']^2+M'),$$

a solution of which is $G(m^3)=\tfrac{1}{3}[m]^3+mM$. Hence we have

$$G(1, m^3)=\tfrac{1}{3}[m]^3+mM+\alpha m+\beta M.$$

Suppose first that the curve m is a system of lines ($m=m$, $M=0$), then $G(1, m^3)=\tfrac{1}{3}[m]^3$; and next that the curve m is a cubic in space or skew cubic ($m=3$, $M=-2$), then $G(1, m^3)=0$, since a line can meet the curve in two points only. We thus find $\alpha=0$, $\beta=-2$, and thence

$$G(1, m^3)=\tfrac{1}{3}[m]^3+M(m-2).$$

33. Hence, substituting for $G(m^3, m')$, $G(m^2, m'^2)$, $G(m, m'^3)$ their values

$m'\left(\tfrac{1}{3}[m]^3+M(m-2)\right)$, $\tfrac{1}{2}[m]^2[m']^2+M.\tfrac{1}{2}[m']^2+M'.\tfrac{1}{2}[m]^2+MM'$, and $m\left(\tfrac{1}{3}[m']^3+M'(m'-2)\right)$

respectively, we find

$$\begin{aligned} G(m+m')^4-G(m^4)-G(m'^4)= \quad & m'\left(\tfrac{1}{3}[m]^3+M(m-2)\right) \\ & +\tfrac{1}{2}[m]^2[m']^2+M.\tfrac{1}{2}[m']^2+M'.\tfrac{1}{2}[m]^2+MM' \\ & + \quad m\left(\tfrac{1}{3}[m']^3+M'(m'-2)\right), \end{aligned}$$

and thence, obtaining first a particular solution, the general solution is

$$G(m^4)=\tfrac{1}{12}[m]^4+M(\tfrac{1}{2}[m]^2-2m)+M^2.\tfrac{1}{2}+\alpha m+\beta M.$$

34. To determine the constants, suppose first that the curve m is a system of lines ($m=m$, $M=0$), we must have $G(m^4)=\frac{1}{12}[m]^4+m$, and thence $\alpha=0$. Next, if the curve m be a conic ($m=2$, $M=-1$), we must have $G(m^4)=0$; and this gives $\beta=\frac{11}{2}$, and consequently

$$G(m^4)=\tfrac{1}{12}[m]^4+m+M(\tfrac{1}{2}[m]^2-2m+\tfrac{11}{2})+M^2.\tfrac{1}{2}.$$

The NG *formulæ*, Article 35.

35. The NG formulæ are now at once obtained, viz. we have

$$\begin{aligned} NG(m, n, p)&=G(m^2, n, p)+G(m, n^2, p)+G(m, n, p^2), \\ NG(m^2, n) &=3G(m^3, n) +G(m^2, n^2), \\ NG(m^3) &=6G(m^3), \end{aligned}$$

which give the values in the Table.

The S *formulæ, particular cases*, Articles 36 to 40.

36. The S formulæ have in fact been obtained in the investigation of the G formulæ: we have

$$\begin{aligned} S(m, n, p)&=2mnp, \\ S(m^2, n) &=n([m]^2+M), \\ S(m^3) &=\tfrac{1}{3}[m]^3+M(m-2). \end{aligned}$$

37. In confirmation of the formula $S(1, m^2)=[m]^2+M$, it is to be remarked that if we take through the line 1 an arbitrary plane, this meets the curve m in m points, and joining these two and two together we have $\frac{1}{2}[m]^2$ lines, each of them meeting the curve m twice and also meeting the line 1; that is, the lines in question are generating lines of the scroll $S(1, m^2)$. The line 1 is, as already mentioned, an $(h=)(\frac{1}{2}[m]^2+M)$tuple line on the scroll; the section by the arbitrary plane is therefore the line 1 taken $(\frac{1}{2}[m]^2+M)$ times, together with the before-mentioned $\frac{1}{2}[m]^2$ lines; that is, the order of the surface is $[m]^2+M$, as it should be. This is in fact the mode in which the order of the scroll $S(1, m^2)$ was originally obtained by Dr. SALMON.

38. As regards the formula $S(m^3)=\frac{1}{3}[m]^3+M(m-2)$, suppose that the curve m is a (p, q) curve on the hyperboloid, we have as before $m=p+q$, $M=-pq$, and the formula becomes

$$S(m^3)=\tfrac{1}{3}[p+q]^3-pq(p+q-2),$$

which is

$$=\tfrac{1}{3}[p]^3+\tfrac{1}{3}[q]^3,$$

viz. as already remarked, the surface is in this case the hyperboloid taken $\frac{1}{6}[p]^3+\frac{1}{6}[q]^3$ times.

39. It is to be noticed also that if the curve m be a system of lines ($m=m$, $\mathrm{M}=0$), then the formula gives

$$\mathrm{S}(m^3)=\tfrac{1}{3}[m]^3,$$

which is right, since in this case the scroll is made up of the $\frac{1}{6}[m]^3$ hyperboloids, generated each of them by a line which meets three out of the m lines.

In the case of a curve m, which is such that the coordinates of any point of the curve are proportional to rational and integral functions of the order m of an arbitrary parameter θ, or say the case of a *unicursal* curve of the order m, we have

$$(h=\tfrac{1}{2}[m-1]^2 \text{ and } \therefore)\mathrm{M}=-(m-1),$$

and the formula gives

$$\mathrm{S}(m^3)=\tfrac{1}{3}[m-1]^3,$$

for a direct investigation of which see *post*, Annex No. 1.

40. In the case of a curve m, which is the complete intersection of two surfaces of the orders p and q respectively, or say a complete $(p\times q)$ intersection, we have

$$m=pq,\ \left(h=\tfrac{1}{2}pq(p-1)(q-1) \text{ and } \therefore\right)\mathrm{M}=-\tfrac{1}{2}pq(p+q-2);$$

and we find

$$\begin{aligned}\mathrm{S}(m^3)&=\tfrac{1}{6}pq(pq-2)(2pq-3p-3q+4)\\&=\tfrac{1}{6}\beta(\beta-2)(2\beta-3\alpha+4)\end{aligned}$$

if $\alpha=pq$, $\beta=p+q$. The mode of obtaining this result by a direct investigation was pointed out to me by Dr. SALMON; see *post*, Annex No. 2.

Particular cases of the formula for G(m^4), Articles 41 & 42.

41. In the case of a (p, q) curve on the hyperboloid, putting as before $m=p+q$, $\mathrm{M}=-pq$, we find

$$\mathrm{G}(m^4)=\tfrac{1}{12}[p+q]^4+p+q-pq\left(\tfrac{1}{2}[p+q]^2-2(p+q)+\tfrac{11}{2}\right)+\tfrac{1}{2}p^2q^2,$$

which is

$$=\tfrac{1}{12}([p]^4+[q]^4)-2q[p-1]^3-2p[q-1]^3,$$

vanishing if p, q are neither of them greater than 3: this is as it should be, since there is then no line which meets the curve four times. The curves for which the condition is satisfied are (1, 1) the conic, (1, 2) the cubic, (2, 2) the quadriquadric, (1, 3) the excubo-quartic, (2, 3) the excubo-quintic (viz. the quintic curve, which is the partial intersection of a quadric surface and a cubic surface having a line in common), and (3, 3) the quadri-cubic, or complete intersection of a quadric surface and a cubic surface. If either p or q exceeds 3, we have the case of a curve through every point whereof there can be drawn a line or lines through four or more points, and the formula is inapplicable.

42. In the case of a complete $(p\times q)$ intersection, we have as before $m=pq$, $M=-\frac{1}{2}pq(p+q-2)$, and the formula for $G(m^4)$ becomes

$$G(m^4)=\tfrac{1}{24}\beta\left\{\begin{array}{l}-66\alpha+144\\ +\beta(3\alpha^2+18\alpha-26)\\ +\beta^2.\,-6\alpha\\ +\beta^3.\,2,\end{array}\right\}$$

a formula the direct verification whereof is due to Dr. SALMON; see *post*, Annex No. 3.

The formulæ for NR(1, m, n) *and* NR(1, m^2), Articles 43 to 46.

43. NR(1, m, n).—Through the line 1 take any plane meeting the curve m in m points and the curve n in n points; then if m_1, m_2 be any two of the m points, and n_1, n_2 any two of the n points, the lines m_1n_1 and m_2n_2 are generating lines of the scroll S(1, m, n), and these lines intersect in a point which belongs to the Nodal Residue NR; and in like manner the lines m_1n_2 and m_2n_1 are generating lines of the scroll, and they intersect on a point of NR; we have thus

$$(2\,.\tfrac{1}{2}[m]^2.\tfrac{1}{2}[n]^2=)\tfrac{1}{2}[m]^2[n]^2$$

points on NR, that is, the arbitrary plane through the line 1 cuts NR in $\frac{1}{2}[m]^2[n]^2$ points. But the plane also cuts NR in certain points lying on the line 1, and if the number of these be (a), then

$$\text{NR}(1,\ m,\ n)=\tfrac{1}{2}[m]^2[n]^2+\text{a}.$$

44. The points (a) are included among the cuspidal points on the line 1. Taking for a moment $x=0$, $y=0$ for the equations of the line 1 (which, as we have seen, is a mn-tuple line on the scroll), the equation of the scroll is of the form $(\text{A},\ldots\between x, y)^{mn}=0$, where A, ... are functions of the coordinates of the degree mn. The entire number of cuspidal points on the line 1 is thus $=2[mn]^2$; but these include different kinds of cuspidal points, viz. we have

$$2[mn]^2=2\text{a}+2\alpha+2\alpha'+\text{R},$$

if (a) be the number of points in which the line 1 meets NR,
,, α ,, ,, ,, ,, S(m^2, n),
,, α' ,, ,, ,, ,, S(m, n^2),
,, R ,, ,, ,, ,, Torse(m, n),

where by Torse(m, n) I denote the developable surface or "Torse" generated by a line which meets each of the curves m and n. The order of the Torse in question is

$$\text{R}=\big(n([m]^2-2h)+m([n]^2-2k)=\big)-2(n\text{M}+m\text{N}),$$

see *post*, Annex No. 4. And then observing that we have

$$\alpha\ =\text{S}(m^2,\ n)=n\,([m]^2+\text{M}),$$
$$\alpha'=\text{S}(m,\ n^2)=m([n]^2+\text{N}\ ),$$

these values give

$$2\alpha+2\alpha'+\mathrm{R}=2n[m]^2+2m[n]^2,$$

and we have

$$\begin{aligned}\mathrm{a}&=\tfrac{1}{2}(2[mn]^2-2\alpha-2\alpha'-\mathrm{R})\\&=[mn]^2-n[m]^2-m[n]^2\\&=[m]^2[n]^2,\end{aligned}$$

and thence

$$\mathrm{NR}(1, m, n)=\tfrac{3}{2}[m]^2[n]^2.$$

45. $\mathrm{NR}(1, m^2)$.—Through the line 1 take any arbitrary plane meeting the curve m in m points; if m_1, m_2, m_3, m_4 be any four of these, then the lines m_1m_2 and m_3m_4 are generating lines of the scroll $\mathrm{S}(1, m^2)$, and their intersection is a point of the nodal residue NR; but in like manner the lines m_1m_3 and m_2m_4 are generating lines of the scroll, and their intersection is a point of NR; and so the lines m_1m_4 and m_2m_3 are generating lines of the scroll, and their intersection is a point of NR. We have thus $(3\times\frac{1}{24}[m]^4=)\frac{1}{8}[m]^4$ points of NR on the arbitrary plane through the line 1. But there are besides the points of NR which lie on the line 1; and if the number of these be (a), then

$$\mathrm{NR}(1, m^2)=\tfrac{1}{8}[m]^4+\mathrm{a}.$$

46. The points (a) are included among the cuspidal points of the scroll lying on the line 1. Supposing for a moment that $x=0$, $y=0$ are the equations of the line 1, then this line being a $(\frac{1}{2}[m]^2+\mathrm{M})$tuple line on the scroll, the equation of the scroll is of the form $(\mathrm{A}, \ldots \between x, y)^{\frac{1}{2}[m]^2+\mathrm{M}}=0$, where A, ... are functions of the coordinates of the degree $\frac{1}{2}[m]^2$: the number of cuspidal points on the line 1 is thus

$$\left(2\cdot\tfrac{1}{2}[m]^2(\tfrac{1}{2}[m]^2-1+\mathrm{M})=\right)[m]^2(\tfrac{1}{2}[m]^2-1+\mathrm{M}).$$

But these include cuspidal points of several kinds, viz., we have

$$[m]^2(\tfrac{1}{2}[m]^2-1+\mathrm{M})=2\mathrm{a}+3\beta+\mathrm{R}'$$

if (a)	be the number of points in which the line 1 meets				NR,
" β	"	"	"	"	$\mathrm{S}(m^3)$,
" R'	"	"	"	"	Torse (m^2),

where Torse (m^2) denotes the developable surface or Torse generated by a line which meets the curve m twice. The order of the Torse in question is

$$\mathrm{R}'=-2(m-3)\mathrm{M},$$

(see *post*, Annex No. 5); and then since

$$\beta=\mathrm{S}(m^3)=\tfrac{1}{3}[m]^3+\mathrm{M}(m-2),$$

we find

$$\begin{aligned}2\mathrm{a}&=[m]^2(\tfrac{1}{2}[m]^2-1+\mathrm{M})-3\left(\tfrac{1}{3}[m]^3+\mathrm{M}(m-2)\right)+2\mathrm{M}(m-3)\\&=\tfrac{1}{2}[m]^4+[m]^3+\mathrm{M}([m]^2-m],\end{aligned}$$

and thence

$$\mathrm{NR}(1, m^2)=\tfrac{3}{8}[m]^4+\tfrac{1}{2}[m]^3+\mathrm{M}(\tfrac{1}{2}[m]^2-\tfrac{1}{2}m).$$

But I have not succeeded in finding by a like direct investigation the values of

$$\mathrm{NR}(m, n, p),\ \mathrm{NR}(m^2, n),\ \mathrm{NR}(m^3).$$

Formulæ for NT(1, m, n), NT(1, m^2), Articles 47 & 48.

47. We have

$$\begin{aligned}\mathrm{NT}(1, m, n) = \quad & \mathrm{NG}(1, m, n) = && mn(m+n-2)+m\mathrm{N}n\mathrm{M} \\ & +\mathrm{ND}(1, m, n) && +\tfrac{1}{2}mn(mn+m+n-3) \\ & +\mathrm{NR}(1, m, n) && +\tfrac{3}{2}[m]^2[n]^2,\end{aligned}$$

which is

$$\begin{aligned}&=2[mn]^2+m\mathrm{N}+n\mathrm{M}\\ &=\tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+m\mathrm{N}+n\mathrm{M},\end{aligned}$$

where $\mathrm{S}=\mathrm{S}(1, m, n)=2mn$.

48. And moreover

$$\begin{aligned}\mathrm{NT}(1, m^2) = \quad & \mathrm{ND}(1, m^2) = && \tfrac{1}{8}[m]^4+[m]^3+\mathrm{M}(\tfrac{1}{2}[m]^2-\tfrac{1}{2})+\mathrm{M}^2\cdot\tfrac{1}{2} \\ & +\mathrm{NG}(1, m^2) && +[m]^3+\mathrm{M}(3m-6) \\ & +\mathrm{NR}(1, m^2) \quad +\tfrac{3}{8}[m]^4 && +\mathrm{M}(\tfrac{1}{2}[m]^2-2m+3),\end{aligned}$$

which is

$$\begin{aligned}&=\tfrac{1}{2}[m]^4+2[m]^3+\mathrm{M}([m]^2+m-\tfrac{7}{2})+\mathrm{M}^2\cdot\tfrac{1}{2}\\ &=\tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+\mathrm{M}(m-\tfrac{5}{2})\end{aligned}$$

if $\mathrm{S}=\mathrm{S}(1, m^2)=[m]^2+\mathrm{M}$.

The NT *and* NR *formulæ*, Articles 49 to 58.

49. I proceed to find NT(m, n, p), &c. by a functional investigation, such as was employed for finding G(1, 1, m^2), &c. Writing S(m) to denote either of the scrolls S(m, n, p), S(m, n^2), and supposing that in place of the curve m we have the aggregate of the two curves m, m'; then the scroll S($m+m'$) breaks up into the scrolls Sm, Sm', and the intersection of these is part of the nodal total NT($m+m'$); that is, we have

$$\mathrm{NT}(m+m')=\mathrm{NT}(m)+\mathrm{NT}(m')+\mathrm{S}(m)\,.\,\mathrm{S}(m');$$

and in like manner, if S(m^2) stands for S(m^2, n), then

$$\mathrm{NT}(m+m')^2=\mathrm{NT}(m^2)+\mathrm{NT}(m, m')+\mathrm{NT}(m'^2)+\mathrm{C}_2\big(\mathrm{S}(m^2), \mathrm{S}(m, m'), \mathrm{S}(m'^2)\big),$$

where C_2 denotes the sum of the combinations two and two together; and so also

$$\begin{aligned}\mathrm{NT}(m+m')^3=&\mathrm{NT}(m^3)+\mathrm{NT}(m^2, m')+\mathrm{NT}(m, m'^2)+\mathrm{NT}(m'^3)\\ &+\mathrm{C}_2\big(\mathrm{S}(m^3), \mathrm{S}(m^2, m'), \mathrm{S}(m, m'^2), \mathrm{S}(m'^3)\big).\end{aligned}$$

50. Instead of assuming

$$\mathrm{NT}=\tfrac{1}{2}\mathrm{S}^2+\varphi,$$

it is the same thing, and it is rather more convenient, to assume

$$\mathrm{NT}=\tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+\varphi,$$

viz. $\mathrm{NT}(m)=\frac{1}{2}\big(\mathrm{S}(m)\big)^2-\mathrm{S}(m)+\phi(m)$, &c. Then observing that

$$\mathrm{S}(m+m')=\mathrm{S}(m)+\mathrm{S}(m'),\ \&c.,$$

the foregoing equations for NT give

$$\begin{aligned}
&\phi(m+m')=\phi(m)+\phi(m'),\\
&\phi(m+m')^2=\phi(m^2)+\phi(m,\ m')+\phi(m'^2),\\
&\phi(m+m')^3=\phi(m^3)+\phi(m^2,\ m')+\phi(m,\ m'^2)+\phi(m'^3);
\end{aligned}$$

and if in the second equation $\phi(m,\ m')$ and in the third equation $\phi(m^2,\ m')$ and $\phi(m,\ m'^2)$ are regarded as known, these are all of them of the form

$$f(m+m')-f(m)-f(m')=\text{Funct.}\ (m,\ m');$$

so that, a particular solution being obtained, the general solution is $f(m)=$ Particular Solution $+\alpha m+\beta\mathrm{M}$, at least on the assumption that $f(m)$, in so far as it depends on the curve m, is a function of m and M only.

51. First, if $\phi(m)$ stands for $\phi(m,\ n,\ p)$, we obtain $\phi(m,\ n,\ p)=\alpha m+\beta\mathrm{M}$, or observing that $\phi(m,\ n,\ p)$ must be symmetrical in regard to the curves m, n, and p, we may write

$$\phi(m,\ n,\ p)=\alpha mnp+\beta(\mathrm{M}np+\mathrm{N}mp+\mathrm{P}mn)+\gamma(m\mathrm{NP}+n\mathrm{MP}+p\mathrm{MN})+\delta\mathrm{MNP},$$

and then

$$\begin{aligned}
\mathrm{NT}(m,\ n,\ p)&=\tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+\phi(m,\ n,\ p)\\
&=2mnp(mnp-1)+\phi(m,\ n,\ p).
\end{aligned}$$

But for $p=1$ this should reduce itself to the known value of $\mathrm{NT}(1,\ m,\ n)$; this gives $\alpha=0$, $\beta=1$, $\gamma=0$; we in fact have, as will be shown, *post*, Art. 55, $\delta=0$; and hence

$$\begin{aligned}
\mathrm{NT}(m,\ n,\ p)&=\tfrac{1}{2}\mathrm{S}^2-\mathrm{S}\ +(\mathrm{M}np+\mathrm{N}mp+\mathrm{P}mn)\\
&=2[mnp]^2+(\mathrm{M}np+\mathrm{N}mp+\mathrm{P}mn).
\end{aligned}$$

52. Next, if $\phi(m^2)$ stand for $\phi(m^2,\ n)$, then $\phi(m,\ m')$ stands for $\phi(m,\ m',\ n)$, which is $=\mathrm{N}mm'+n(m\mathrm{M}'+m'\mathrm{M})$, and the equation is

$$\phi(m+m')^2-\phi(m^2)-\phi(m'^2)=\mathrm{N}mm'+n(m\mathrm{M}'+m'\mathrm{M}).$$

A particular solution is $\phi(m^2)=\frac{1}{2}[m]^2\mathrm{N}+nm\mathrm{M}$, and we have therefore

$$\phi(m^2,\ n)=\tfrac{1}{2}[m]^2\mathrm{N}+nm\mathrm{M}+\alpha m+\beta\mathrm{M};$$

or observing that $\phi(m^2,\ n)$ considered as a function of n, satisfies the equation

$$\phi(n+n')=\phi(n)+\phi(n'),$$

and is therefore a linear function of n and N, we may write

$$\phi(m^2,\ n)=\tfrac{1}{2}[m]^2\mathrm{N}+nm\mathrm{M}+\alpha nm+\beta n\mathrm{M}+\gamma m\mathrm{N}+\delta\mathrm{MN};$$

we then have

$$\mathrm{NT}(m^2,\ n)=\tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+\phi(m^2,\ n),$$

where

$$\mathrm{S}=\mathrm{S}(m^2,\ n)=n([m]+\mathrm{M}).$$

And then putting $n=1$, and comparing with the known value of $\mathrm{NT}(1, m^2)$, we find $\alpha=0$, $\beta=-\frac{5}{2}$. It will be shown, *post*, Art. 55, that $\gamma=0$, $\delta=0$; and we have therefore

$$\phi(m^2, n)=n\mathrm{M}(m-\tfrac{5}{2})+\mathrm{N}(\tfrac{1}{2}[m]^2+\mathrm{M}),$$

and thence

$$\begin{aligned}\mathrm{NT}(m^2, n)&=\tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+\phi(m^2, n)\\ &=n\left(\tfrac{1}{2}[m]^4+2[m]^3+\mathrm{M}([m]^2+m-\tfrac{7}{2})+\mathrm{M}^2\,.\,\tfrac{1}{2}\right)\\ &\quad+[n]^2(\tfrac{1}{2}[m]^4+2[m]^3+[m]^2+\mathrm{M}[m]^2+\mathrm{M}^2\,.\,\tfrac{1}{2})\\ &\quad+\mathrm{N}(\tfrac{1}{2}[m]^2+\mathrm{M}).\end{aligned}$$

53. Next for $\phi(m^3)$, substituting for $\phi(m^2, m')$ and $\phi(m, m'^2)$ their values, we have

$$\begin{aligned}\phi(m+m')^3-\phi(m^3)-\phi(m'^3)&=m'\mathrm{M}(m-\tfrac{5}{2})+\mathrm{M}'(\tfrac{1}{2}[m]^2+\mathrm{M})\\ &\quad+m\mathrm{M}'(m'-\tfrac{5}{2})+\mathrm{M}(\tfrac{1}{2}[m']^2+\mathrm{M}'),\end{aligned}$$

which is satisfied by

$$\phi(m^3)=\mathrm{M}(\tfrac{1}{2}[m]^2-\tfrac{5}{2}m)+\mathrm{M}^2,$$

and the general value then is

$$\phi(m^3)=\mathrm{M}(\tfrac{1}{2}[m]^2-\tfrac{5}{2}m)+\mathrm{M}^2+\alpha m+\beta\mathrm{M},$$

and we have

$$\mathrm{NT}(m^3)=\tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+\phi(m^3),$$

where

$$\mathrm{S}=\mathrm{S}(m^3)=\tfrac{1}{3}[m]^3+\mathrm{M}(m-2).$$

54. Taking for the curve m the (p, q) curve on the hyperboloid ($m=p+q$, $\mathrm{M}=-pq$), $\mathrm{S}(m^3)$ becomes the hyperboloid taken k times, if $k=\frac{1}{6}[p]^3+\frac{1}{6}[q]^3$; that is, $\mathrm{S}(m^3)=2k$, and $\mathrm{NT}(m^3)=4\,.\,\frac{1}{2}[k]^2+\phi(m^3)$; $\phi(m^3)$ must vanish if p and q are each not greater than 3, this implies $\alpha=3$, $\beta=11$, for with these values the formula gives

$$\phi(m^3)=-\tfrac{1}{2}(q[p-1]^3+p[q-1]^3).$$

55. I assume the correctness of the value

$$\phi(m^3)=3m+\mathrm{M}(\tfrac{1}{2}[m]^2-\tfrac{5}{2}m+11)+\mathrm{M}^2$$

so obtained, as being in fact verified by means of the six several curves (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3); and I remark that if the foregoing value of $\phi(m, n, p)$ had been increased by $6\alpha\mathrm{MNP}$, then it would have been necessary to increase the value of $\phi(m^2, n)$ by $3\alpha\mathrm{M}^2\mathrm{N}$, and that of $\phi(m^3)$ by $\alpha\mathrm{M}^3$; and moreover that if the foregoing value of $\phi(m^2, n)$ had been increased by $\gamma m\mathrm{N}+\delta\mathrm{MN}$, then it would have been necessary to increase the value of $\phi(m^3)$ by $\gamma m\mathrm{M}+\delta\mathrm{M}^2$; this is easily seen by writing down the values

$$\begin{aligned}\phi(m^3)\quad&=\gamma m\mathrm{M}+\delta\mathrm{M}^2+\ \alpha\mathrm{M}^3\ ,\\ \phi(m^2, m')&=\gamma m\mathrm{M}'+\delta\mathrm{MM}'+3\alpha\mathrm{M}^2\mathrm{M}',\\ \phi(m, m'^2)&=\gamma m'\mathrm{M}+\delta\mathrm{MM}'+3\alpha\mathrm{MM}'^2,\\ \phi(m'^3)\quad&=\gamma m'\mathrm{M}'+\delta\mathrm{M}'^2+\ \alpha\mathrm{M}'^3\ ,\end{aligned}$$

the sum of which is

$$=\gamma(m+m')(\mathrm{M}+\mathrm{M}')+\delta(\mathrm{M}+\mathrm{M}')^2+\alpha(\mathrm{M}+\mathrm{M}')^3,$$

the corresponding term of $\phi(m^3)$; hence the value of $\phi(m^3)$ being correct without the foregoing addition, we must have $\gamma=0$, $\delta=0$, $\alpha=0$; which confirms the foregoing values of $\phi(m, n, p)$, $\phi(m^2, n)$.

56. The equation

$$\mathrm{NT}(m^3)=\tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+\phi(m^3)$$

gives

$$\begin{aligned}\mathrm{NT}(m^3)&=\tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+3m+\mathrm{M}(\tfrac{1}{2}[m]^2-\tfrac{5}{2}m+11)+\mathrm{M}^2\\&=\tfrac{1}{18}[m]^6+\tfrac{1}{2}[m]^5+[m]^4+3m\\&\quad+\mathrm{M}(\tfrac{1}{3}[m]^4+\tfrac{1}{3}[m]^3+\tfrac{1}{2}[m]^2-\tfrac{7}{2}m+13)\\&\quad+\mathrm{M}^2(\tfrac{1}{2}[m]^2-\tfrac{3}{2}m+3).\end{aligned}$$

57. We have

$$\begin{aligned}\mathrm{NR}(m^2, n)=&\ \ \mathrm{NT}(m^2, n)-\mathrm{ND}(m^2, n)-\mathrm{NG}(m^2, n)\\=&\ \ n\ \left(\tfrac{3}{8}[m]^4+\mathrm{M}(\tfrac{1}{2}[m]^2-2m+3)\right)\\&+[n]^2\left(\tfrac{1}{2}[m]^4+\tfrac{3}{2}[m]^3+[m]^2+\mathrm{M}([m]^2-\tfrac{1}{2})+\mathrm{M}^2.\tfrac{1}{2}\right).\end{aligned}$$

58. And moreover,

$$\begin{aligned}\mathrm{NR}(m^3)=&\ \ \mathrm{NT}(m^3)-\mathrm{ND}(m^3)-\mathrm{NG}(m^3)\\=&\ \ \tfrac{1}{18}[m]^6+\tfrac{3}{8}[m]^5-\tfrac{1}{2}[m]^3+3m\\&+\mathrm{M}(\tfrac{1}{3}[m]^4-\tfrac{1}{6}[m]^3-\tfrac{5}{2}[m]^2+8m-20)+\mathrm{M}^2(\tfrac{1}{2}[m]^2-2m):\end{aligned}$$

and the investigation of the series of results given in the Table is thus concluded.

Intersections of a generating line with the Nodal Total, Articles 59 to 63.

59. We may for the scrolls $\mathrm{S}(1, m, n)$ and $\mathrm{S}(1, m^2)$ verify the theorem that each generating line meets the Nodal Total in a number of points$=\mathrm{S}-2$.

In fact for the scroll $\mathrm{S}(1, m, n)$, the directrix curves are respectively multiple curves of the orders mn, n, m, and a generating line meets each of these in a single point, counting for the three curves respectively as $mn-1$, $n-1$, and $m-1$ points respectively. Moreover the construction (*ante*, Art. 43) for the Nodal Residue $\mathrm{NR}(1, m, n)$ shows that a generating line meets this curve in $(m-1)(n-1)$ points; and since the curve is merely a double curve, these count each as a single point; and the generating line does not meet the Nodal Generator $\mathrm{NG}(1, m, n)$. The number of intersections therefore is

$$mn-1+(m-1)+(n-1)+(m-1)(n-1),$$

which is

$$=2mn-2,\quad =\mathrm{S}-2.$$

60. Similarly for the scroll $\mathrm{S}(1, m^2)$; the directrix curves are multiple curves, viz. the line 1 is a $(\tfrac{1}{2}[m]^2+\mathrm{M})$tuple curve, and the curve m a $(m-1)$tuple curve; the

generating line meets the former in a single point, counting as $\frac{1}{2}[m]^2+\text{M}-1$ points, and the latter in two points, each counting as $(m-2)$ points. The construction (*ante*, Art. 45) for the Nodal Residue NR$(1, m^2)$ shows that the generating line meets this curve in $\frac{1}{2}[m-2]^2$ points; and since the curve is merely a double curve, these count each as a single point. Finally, the generating line does not meet the Nodal Generator NG$(1, m^2)$. The number of intersections thus is

$$\tfrac{1}{2}[m]^2-1+\text{M}+2(m-2)+\tfrac{1}{2}[m-2]^2,$$

which is

$$=[m]^2-2+\text{M}, \quad =\text{S}-2.$$

In the remaining cases we may use the theorem to find the number of points in which the generating line meets the Nodal Residue. Using Π as the symbol for the points in question $\big(\Pi(m, n, p)$ for the scroll $\text{S}(m, n, p)$, &c.$\big)$, we find

61. For the scroll $\text{S}(m, n, p)$,

$$(mn-1)+(np-1)+(mp-1)+\Pi(m, n, p)=\text{S}-2=2mnp-2,$$

which gives

$$\Pi(m, n, p)=2mnp-mn-mp-np+1.$$

This includes the before-mentioned case

$$\Pi(1, m, n)=(m-1)(n-1),$$

and the more particular one

$$\Pi(1, 1, m)=0.$$

62. For the scroll $\text{S}(m^2, n)$,

$$\tfrac{1}{2}[m]^2-1+\text{M}+2\big((m-1)n-1\big)+\Pi(m^2, n)=\text{S}-2=n([m]^2+\text{M})-2,$$

which gives

$$\begin{aligned}\Pi(m^2, n)=n([m]^2&-2m+2+\text{M})\\ &-\tfrac{1}{2}[m]^2+1-\text{M}.\end{aligned}$$

This includes the before-mentioned particular case

$$\Pi(1, m^2)=\tfrac{1}{2}[m-2]^2.$$

63. And lastly for the scroll $\text{S}(m^3)$,

$$3(\tfrac{1}{2}[m]^2-m+1+\text{M})+\Pi(m^3)=\text{S}-2=\tfrac{1}{3}[m]^3+(m-2)\text{M}-2,$$

which gives

$$\Pi(m^3)=\tfrac{1}{3}[m]^3-\tfrac{3}{2}[m]^2+3m-5+\text{M}(m-5).$$

The foregoing expressions for Π might with propriety have been inserted in the Table.

Annex No. 1.—*Investigation of the formula for* $\text{S}(m^3)$ *in the case of the unicursal curve* (referred to, Art. 39).

Consider the unicursal m-thic curve the equations whereof are $x:y:z:w=\text{A}:\text{B}:\text{C}:\text{D}$, where A, B, C, D are rational and integral functions of a parameter θ. And let it be

required to find the equation of a plane meeting the curve in such manner that three of the points of intersection are *in lineâ*. Taking for the equation of the plane

$$\xi x+\eta y+\zeta z+\omega w=0,$$

we find between $(\xi, \eta, \zeta, \omega)$ an equation of a certain degree in $(\xi, \eta, \zeta, \omega)$, which is the equation in plane-coordinates of the scroll $S(m^3)$, the degree of the equation is therefore equal to the class of the scroll; but as the class of a scroll is equal to its order, the degree of the equation is equal to the order of the scroll, or say $=S(m^3)$.

Proceeding with the investigation, if θ be determined by the equation

$$\xi A+\eta B+\zeta C+\omega D=0,$$

then the roots $\theta_1, \theta_2, \ldots \theta_m$ of this equation belong to the points of intersection of the plane and curve; and the corresponding coordinates of these points are (A_1, B_1, C_1, D_1), &c.

Suppose that the points 1, 2, 3 are *in lineâ*, and let $\lambda, \mu, \nu, \varrho$ be the coordinates of an arbitrary point, then the four points are *in plano*, that is, we have

$$\begin{vmatrix} \lambda, & \mu, & \nu, & \varrho \\ A_1, & B_1, & C_1, & D_1 \\ A_2, & B_2, & C_2, & D_2 \\ A_3, & B_3, & C_3, & D_3 \end{vmatrix}=0;$$

and if we form the equation

$$\Pi\begin{vmatrix} \lambda, & \mu, & \nu, & \varrho \\ A_1, & B_1, & C_1, & D_1 \\ A_2, & B_2, & C_2, & D_2 \\ A_3, & B_3, & C_3, & D_3 \end{vmatrix}=0,$$

where Π denotes the product of the terms belonging to all the triads of the m roots, the result will be symmetrical in regard to all the roots; and replacing the symmetrical functions of the roots by their values in terms of the coefficients, we have the required relation between $(\xi, \eta, \zeta, \omega)$.

Π contains $\frac{1}{6}[m]^3$ terms, whereof $\frac{1}{2}[m-1]^2$ contain the m-thic functions (A_1, B_1, C_1, D_1) of the root θ_1; that is, the form of Π is

$$(\lambda, \mu, \nu, \varrho)^{\frac{1}{6}[m]^3}(\theta_1, 1)^{\frac{1}{2}[m]^3}(\theta_2, 1)^{\frac{1}{2}[m]^3}\ldots;$$

or, when the symmetrical functions are expressed in terms of the coefficients, the form is

$$(\lambda, \mu, \nu, \varrho)^{\frac{1}{6}[m]^3}(\xi, \eta, \zeta, \omega)^{\frac{1}{2}[m]^3}.$$

Now the above-mentioned determinant is divisible by $(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_2-\theta_3)$, or Π is divisible by $\Pi(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_2-\theta_3)$; and since this product contains $(3\times\frac{1}{6}[m]^3=)\frac{1}{2}[m]^3$ linear factors, and the product $\zeta(\theta_1, \theta_2, \ldots \theta_m)$ of the squared differences of the roots contains $(2\times\frac{1}{2}[m]^2=)[m]^2$ linear factors, so that we have

$$\Pi(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_2-\theta_3)=\{\zeta(\theta_1, \theta_2, \ldots \theta_m)\}^{\frac{1}{2}(m-2)},$$

where

$$\zeta(\theta_1, \theta_2, .. \theta_m)=\text{Disct.}=(\xi, \eta, \zeta, \omega)^{2(m-1)},$$

and consequently

$$\Pi(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_2-\theta_3)=(\xi, \eta, \zeta, \omega)^{[m-1]^2},$$

so that, omitting this factor, the remaining factor of Π is of the form

$$(\lambda, \mu, \nu, \varrho)^{\frac{1}{6}[m]^3}(\xi, \eta, \zeta, \omega)^{\frac{1}{2}[m]^3-[m-1]^2};$$

but the determinant vanishes if

$$\lambda, \mu, \nu, \varrho=(A_1, B_1, C_1, D_1), \quad (A_2, B_2, C_2, D_2), \quad (A_3, B_3, C_3, D_3),$$

or say if

$$(\lambda, \mu, \nu, \varrho)=(A, B, C, D), \quad \theta=\theta_1, \theta_2, \text{ or } \theta_3;$$

it follows that the product Π contains the factor

$$(\lambda\xi+\mu\eta+\nu\zeta+\varrho\omega)^{\frac{1}{6}[m]^3};$$

or omitting this factor, and observing that

$$\tfrac{1}{2}[m]^3-[m-1]^2-\tfrac{1}{6}[m]^3=\tfrac{1}{3}[m]^3-[m-1]^2=\tfrac{1}{3}[m-1]^3,$$

the remaining factor is of the form

$$(\xi, \eta, \zeta, \omega)^{\frac{1}{3}[m-1]^3};$$

or we have finally

$$S(m^3)=\tfrac{1}{3}[m-1]^3,$$

which is the required expression.

I give the following investigation of the expression $\frac{1}{2}[m-1]^2$ for the number of apparent double points. Imagine through the point ($x=0$, $y=0$, $z=0$) a line cutting the curve in the two points corresponding to the values θ_1, θ_2 of the parameter. We have

$$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2},$$

which equations determine θ_1 and θ_2.

Writing the equations under the form

$$\frac{A_1B_2-A_2B_1}{\theta_1-\theta_2}=0, \quad \frac{A_1C_2-A_2C_1}{\theta_1-\theta_2}=0,$$

and treating θ_1 and θ_2 as coordinates, each of these equations belongs to a curve of the order $2(m-1)$, having a $(m-1)$thic point at infinity on each of the axes. The number of intersections thus is

$$=4(m-1)^2-(m-1)^2-(m-1)^2, \quad =2(m-1)^2.$$

But among these are included points not belonging to the original system, viz. the points for which ($A_1=0$, $A_2=0$) other than those for which $\theta_1=\theta_2$; the points so included are in number $=m^2-m$; and omitting them, the number is

$$\big(2(m-1)^2-m(m-1)\big)=[m-1]^2,$$

which is the number of points θ_1 lying *in lineâ* with the origin and another point θ_2; the number of apparent double points is the half of this, or $h=\frac{1}{2}[m-1]^2$. And thence

$$M=(-\tfrac{1}{2}[m]^2+h=)-(m-1).$$

I investigate also the number of lines through two points which meet two arbitrary lines; this is in fact $=S(1, m^2)$, which for the curve in question is

$$=\left(\tfrac{1}{2}[m]^2-(m-1)=\right)(m-1)^2.$$

Let the equations of the two lines be $(x=0, y=0)$ and $(z=0, w=0)$; then the conditions to be satisfied are

$$\frac{A_1}{A_2}=\frac{B_1}{B_2},\quad \frac{C_1}{C_2}=\frac{D_1}{D_2};$$

or writing these under the form

$$\frac{A_1B_2-A_2B_1}{\theta_1-\theta_2}=0,\quad \frac{C_1D_2-C_2D_1}{\theta_1-\theta_2}=0,$$

and treating θ_1, θ_2 as coordinates, the number of intersections of these two curves is $=2(m-1)^2$, the same as in the two curves last above considered. And the number of the lines in question is one half of this, or $=(m-1)^2$.

Lemma employed in the following Annexes 2 and 3. *Formulæ for the order and weight of certain systems of equations.*

Let $\alpha_{\alpha'}$ denote a function of the degree α in the *order* variables $(x, y, ..)$, and of the degree α' in the *weight* variables $(x', y', ..)$, and so in other cases; and consider first the equation

$$\begin{vmatrix} \alpha_{\alpha'}, & (\alpha+A)_{\alpha'+A'}, & \ldots \\ \beta_{\beta'}, & (\beta+A)_{\beta'+A'}, & \\ \vdots & & \end{vmatrix}=0,$$

where the matrix is a square; then

$$\begin{aligned} \text{Order} &=\Sigma\alpha+\Sigma A, \\ \text{Weight} &=\Sigma\alpha'+\Sigma A'. \end{aligned}$$

Consider next the system

$$\begin{Vmatrix} \alpha_{\alpha'}, & (\alpha+A)_{\alpha'+A'}, & (\alpha+B)_{\alpha'+B'}, & \ldots \\ \beta_{\beta'}, & (\beta+A)_{\beta'+A'}, & (\beta+B)_{\beta'+B'}, & \\ \vdots & & & \end{Vmatrix}=0,$$

where the matrix is a square $+1$, that is, the number of columns exceeds by 1 the number of lines; then

$$\begin{aligned} \text{Order} &=\Sigma AB-\Sigma\alpha\beta+\Sigma\alpha(\Sigma A+\Sigma\alpha), \\ \text{Weight} &=(\Sigma A+\Sigma\alpha)(\Sigma A'+\Sigma\alpha')-\Sigma AA'+\Sigma\alpha\alpha'. \end{aligned}$$

And again, the system

$$\left\|\begin{array}{l} \alpha_{\alpha'},\ (\alpha+A)_{\alpha'+A'},\ (\alpha+B)_{\alpha'+B'},\ (\alpha+C)_{\alpha'+C'},\ \ldots \\ \beta_{\beta'},\ (\beta+A)_{\beta'+A'},\ (\beta+B)_{\beta'+B'},\ (\beta+C)_{\beta'+C'}, \\ \cdot \\ \cdot \\ \cdot \end{array}\right\| = 0,$$

where the matrix is a square $+2$, that is, the number of columns exceeds by 2 the number of lines; then

$$\begin{aligned} \text{Order } &= \Sigma ABC+\Sigma\alpha\beta\gamma+\Sigma\alpha(\Sigma AB-\Sigma\alpha\beta)+\big((\Sigma\alpha)^2-\Sigma\alpha\beta\big)(\Sigma A+\Sigma\alpha), \\ \text{Weight} &= \{\Sigma AB-\Sigma\alpha\beta+\Sigma\alpha(\Sigma A+\Sigma\alpha)\}(\Sigma A'+\Sigma\alpha')-(\Sigma A+\Sigma\alpha)(\Sigma AA'-\Sigma\alpha\alpha') \\ &\quad +\Sigma A^2A'+\Sigma\alpha^2\alpha'. \end{aligned}$$

The last formula, for the weight of the square $+2$ system, was communicated to me by Dr. SALMON, the others are all in effect given in the Appendix, "On the Order of Systems of Equations," to his Treatise on the Analytic Geometry of Three Dimensions; and in the investigation in the following Annexes 2 and 3, the route which I have followed was completely traced out for me by him, so that I have only supplied the details of the work.

Annex No. 2.—*Investigation of the formula for* $S(m^3)$, *when the curve* m *is the* pq *complete intersection, viz. when it is the intersection of two surfaces of the orders* p *and* q *respectively* (referred to, Art. 40).

Let $U=0$, $V=0$ be the equations of the two surfaces of the orders p and q respectively. Take (x, y, z, w) the coordinates of a point on the curve, so that for these coordinates we have $U=0$, $V=0$; and in the equations of the two curves respectively, write for the coordinates $x+\varrho x'$, $y+\varrho y'$, $z+\varrho z'$, $w+\varrho w'$; then putting for shortness

$$\Delta=x'\partial_x+y'\partial_y+z'\partial_z+w'\partial_w,$$

the resulting equations may be represented by

$$\begin{aligned} (\Delta U,\ \Delta^2U,\ \ldots\ \Delta^pU \between 1,\ \varrho)^{p-1}&=0, \\ (\Delta V,\ \Delta^2V,\ \ldots\ \Delta^qV \between 1,\ \varrho)^{q-1}&=0, \end{aligned}$$

where it is to be noticed that besides the expressed literal coefficients there are numerical coefficients (not as the notation usually denotes, the binomial coefficients, but) $=\frac{1}{1}, \frac{1}{1.2}, \frac{1}{1.2.3}$, &c.

Supposing that (x', y', z', w') are the current coordinates of a point on the line drawn through the point (x, y, z, w) to meet the curve in two other points, the equations in ϱ must have two common roots, and this gives a system equivalent to two equations, or say a plexus of two equations. If from the plexus and the two equations $U=0$, $V=0$ we eliminate (x, y, z, w), we obtain an equation $S'=0$ in (x', y', z', w'), which is in fact the equation of the scroll $S(m^3)$, taken (as is easily seen to be the case) thrice; that is, $S(m^3)=\frac{1}{3}$Degree of S'. But observing that the coordinates (x', y', z', w') enter into the plexus only

and not into the functions U, V, and treating (x', y', z', w') as *weight* variables, Degree of S'=Weight of System (U=0, V=0, Plexus)=Deg. U×Deg. V×Weight of Plexus, $=pq\times$Weight of Plexus; or, writing $pq=\beta$,

$$S(m^3)=\tfrac{1}{3}\beta\times\text{Weight of Plexus}.$$

The plexus in question is the square +1 system,

$$\left\|\begin{array}{l} \Delta U,\ \Delta^2 U,\ldots \\ \quad\ \Delta U,\ldots \\ \cdot \\ \cdot \\ \Delta V,\ \Delta^2 V,\ldots \\ \quad\ \Delta V,\ldots \\ \cdot \\ \cdot \end{array}\right\|=0,$$

$p+q-3$ columns, $(q-2)+(p-2)=(p+q-4)$ lines; or representing the terms according to their order and weight, that is, degree in (x, y, z, w) and (x', y', z', w') respectively (the order and weight of the evanescent terms being fixed so as that they may form a regular series with the other terms), the system is

$$\left\|\begin{array}{ll} & \overbrace{\qquad\qquad\qquad\qquad}^{p+q-3\text{ columns.}} \\ (q-2)\text{ lines.}\left\{\begin{array}{l}\\ \\ \\ \\ \end{array}\right. & \begin{array}{l} (p-1)_1,\ (p-2)_2,\ \ldots \\ \ p_0\quad,\ (p-1)_1, \\ \ \cdot \\ \ \cdot \end{array} \\ (p-2)\text{ lines.}\left\{\begin{array}{l}\\ \\ \\ \\ \end{array}\right. & \begin{array}{l} (q-1)_1,\ (q-2)_2, \\ \ q_0\quad,\ (q-1)_1, \\ \ \cdot \\ \ \cdot \end{array} \end{array}\right\|=0,$$

so that

$$\begin{array}{llllll} \alpha\,,\beta\,,\ldots= & p-1,\ p,\ \ldots\ p+q-4, & q-1,\ q,\ \ldots & p+q-4, \\ \alpha'\,,\beta',\ldots= & \quad 1,\ 0,\ \ldots\ -q+4, & \quad 1,\ 0,\ \ldots & -p+4, \\ A,\ B,\ldots= & -1,-2, & & \ldots-(p+q-4), \\ A',\ B',\ldots= & \quad 1,\ 2, & & p+q-4, \end{array}$$

or, as regards the first two lines,

$$\left.\begin{array}{ll} \alpha,\ \beta,\ldots=p-2+\theta, & q-2+\phi \\ \alpha',\ \beta',\ldots=\quad 2-\theta, & \quad 2-\phi \end{array}\right\}\theta=1\text{ to }q-2,\text{ and }\phi=1\text{ to }p-2.$$

We then find

$$\begin{aligned} \Sigma\alpha= &\ (q-2)(p-2)+\tfrac{1}{2}(q-2)(q-1) \\ &+(p-2)(q-2)+\tfrac{1}{2}(p-2)(p-1), \\ \Sigma\alpha'= &\ 2(q-2)-\tfrac{1}{2}(q-2)(q-1) \\ &+2(p-2)-\tfrac{1}{2}(p-2)(p-1), \end{aligned}$$

$$\Sigma A \;=-\Sigma A'=-\tfrac{1}{2}(p+q-4)(p+q-3),$$

$$\Sigma\alpha\alpha' = \;2(p-2)(q-2)-(p-4)\,.\,\tfrac{1}{2}(q-2)(q-1)-\tfrac{1}{6}(q-2)(q-1)(2q-3)$$
$$+2(q-2)(p-2)-(q-4)\,.\,\tfrac{1}{2}(p-2)(p-1)-\tfrac{1}{6}(p-2)(p-1)(2p-3),$$

$$\Sigma AA'=-\tfrac{1}{6}(p+q-4)(p+q-3)(2p+2q-7),$$

which putting therein $p+q=\alpha$, $pq=\beta$, give

$$\begin{aligned}
\Sigma\alpha &= \beta+\tfrac{1}{2}\alpha^2-\tfrac{11}{2}\alpha+10,\\
\Sigma\alpha' &= \beta-\tfrac{1}{2}\alpha^2+\tfrac{7}{2}\alpha-10,\\
\Sigma A &=-\Sigma A'= -\tfrac{1}{2}\alpha^2+\tfrac{7}{2}\alpha-6,\\
\Sigma\alpha\alpha' &= \tfrac{1}{2}\alpha\beta-\tfrac{1}{3}\alpha^3+\tfrac{7}{2}\alpha^2-\tfrac{103}{6}\alpha+26,\\
\Sigma AA' &= -\tfrac{1}{3}\alpha^3+\tfrac{7}{2}\alpha^2-\tfrac{73}{6}\alpha+14,
\end{aligned}$$

and thence

$$\begin{aligned}
\Sigma A+\Sigma\alpha &= \beta-2\alpha+4,\\
\Sigma A'+\Sigma\alpha' &= \beta\quad -4,\\
\Sigma AA'-\Sigma\alpha\alpha' &= -\tfrac{1}{2}\alpha\beta+5\alpha-12,
\end{aligned}$$

and therefore

$$\begin{aligned}
\text{Weight} &= (\beta-2\alpha+4)(\beta-4)+\tfrac{1}{2}\alpha\beta-5\alpha+12\\
&= \beta^2-\tfrac{3}{2}\alpha\beta+6\alpha-8\\
&=\tfrac{1}{2}(\beta-2)(2\beta-3\alpha+4),
\end{aligned}$$

and consequently

$$\begin{aligned}
S(m^3)&=\tfrac{1}{3}\beta\times\text{weight}\\
&=\tfrac{1}{6}\beta(\beta-2)(2\beta-3\alpha+4),
\end{aligned}$$

which is right.

Annex No. 3.—*Investigation of* $G(m^4)$ *in the case where the curve m is a pq complete intersection* (referred to, Art. 42).

Suppose, as before, that $U=0$, $V=0$ are the equations of the two surfaces of the orders p and q respectively; taking also (x, y, z, w) as the coordinates of a point on the curve, and substituting in the equations $x+\varrho x'$, $y+\varrho y'$, $z+\varrho z'$, $w+\varrho w'$ in place of the coordinates, then if $\Delta=x'\partial_x+y'\partial_y+z'\partial_z+w\partial_w$, we have as before

$$(\Delta U, \Delta^2 U, \ldots \Delta^p U ⟆ 1, \varrho)^{p-1}=0,$$
$$(\Delta V, \Delta^2 V, \ldots \Delta^q V ⟆ 1, \varrho)^{q-1}=0,$$

where the numerical coefficients $\frac{1}{1}$, $\frac{1}{1.2}$, $\frac{1}{1.2.3}$, &c. are to be understood as before.

Suppose now that (x, y, z, w) are the coordinates of a point on the curve, through which point there passes a line through three other points, or line $G(m^4)$; and that (x', y', z', w') are the current coordinates of a point on such line: the two equations in ϱ must have three equal roots; or we must have a system equivalent to three equations, or say a plexus of three equations. The coordinates (x', y', z', w'), although four in

number, are in fact eliminable from this plexus; or what is the same thing, combining with the plexus the equation

$$\alpha x'+\beta y'+\gamma z'+\delta w'=0$$

of an arbitrary plane, and then eliminating (x', y', z', w'), the result is of the form

$$(\alpha x+\beta y+\gamma z+\delta w)^{\theta}\square=0,$$

where $\square$ is a function of (x, y, z, w) only; and considering (x, y, z, w) as weight variables, θ=Order of Plexus. But degree in (x, y, z, w) of $(\alpha x+\beta y+\gamma z+\delta w)^{\theta}\square$ is =Weight of Plexus, and therefore Degree of $\square$ is = Weight of Plexus$-\theta$, =(Weight$-$Order) of Plexus.

The equations $\mathrm{U}=0$, $\mathrm{V}=0$, $\square=0$ then give the coordinates (x, y, z, w) of the points through which may be drawn a line $\mathrm{G}(m^4)$; viz. they give (as it is easy to see) these points four times over. And we therefore have

$$\begin{aligned}\mathrm{G}(m^4)&=\tfrac{1}{4}\text{ Order of }(\mathrm{U}=0,\ \mathrm{V}=0,\ \square=0)\\&=\tfrac{1}{4}\text{ Deg. U. Deg. V. Deg. }\square\\&=\tfrac{1}{4}\beta\times(\text{Weight}-\text{Order})\text{ of Plexus.}\end{aligned}$$

The Plexus is here the square +2 system

$$\left\|\begin{array}{ll}\Delta\mathrm{U}, & \Delta^2\mathrm{U}, \ldots\\ . & \Delta\,\mathrm{U},\\ . & \\ . & \\ \Delta\mathrm{V}, & \Delta^2\mathrm{V},\\ & \Delta\,\mathrm{V},\\ . & \\ . & \end{array}\right\|=0,$$

$\big(p+q-4$ columns, $(q-3)+(p-3)=p+q-6$ lines$\big)$. Or representing the terms by their order and weight $\big($the weight variables being in the present case (x, y, z, w), and the order variables $(x', y', z', w')\big)$, and attributing as before an order and weight to the evanescent terms, the system is

$$\left\|\begin{array}{l}\overbrace{\qquad\qquad\qquad\qquad}^{p+q-3\text{ columns.}}\\ q-3\text{ lines.}\left\{\begin{array}{ll}1_{p-1}, & 2_{p-2}, \ldots\\ 0_p\ \ , & 1_{p-1},\\ . & \\ . & \end{array}\right.\\ p-3\text{ lines.}\left\{\begin{array}{ll}1_{q-1}, & 2_{q-2},\\ 0_q\ \ , & 1_{q-1},\\ . & \\ . & \end{array}\right.\end{array}\right\|=0,$$

so that we have

$$\begin{array}{llll}\alpha, \beta, \ldots = & 1, 0, -1, \ldots -(q-5), & 1, 0, -1, & \ldots -(p-5),\\ \alpha', \beta', \ldots = & p-1, p, p+1, \ldots p+q-5, & q-1, q, q+1, & \ldots q+p-5,\\ \mathrm{A}, \mathrm{B}, \ldots = & 1, \quad 2, \ldots & & p+q-5,\\ \mathrm{A}', \mathrm{B}', \ldots = & -1, -2, \ldots & & -(p+q-5),\end{array}$$

or, as regards the first two lines,

$$\left.\begin{array}{l}\alpha, \beta, \ldots = \quad 2-\theta, 2-\phi\\ \alpha', \beta', \ldots = p-2+\theta, p-2+\phi\end{array}\right\}\theta=1 \text{ to } q-3,\ \phi=1 \text{ to } p-3.$$

We then find

$$\begin{aligned}
\Sigma\alpha &= 2(q-3)-\tfrac{1}{2}(q-3)(q-2)\\
&\quad+2(p-3)-\tfrac{1}{2}(p-3)(p-2),\\
\Sigma\alpha' &= (p-2)(q-3)+\tfrac{1}{2}(q-3)(q-2)\\
&\quad+(q-2)(p-3)+\tfrac{1}{2}(p-3)(p-2),\\
\Sigma\alpha^2 &= 4(q-3)-4\,.\,\tfrac{1}{2}(q-3)(q-2)+\tfrac{1}{6}(q-3)(q-2)(2q-5)\\
&\quad+4(p-3)-4\,.\,\tfrac{1}{2}(p-3)(p-2)+\tfrac{1}{6}(p-3)(p-2)(2p-5),\\
\Sigma\alpha^3 &= 8(q-3)-12\,.\,\tfrac{1}{2}(q-3)(q-2)+6\,.\,\tfrac{1}{6}(q-3)(q-2)(2q-5)-\tfrac{1}{4}(q-3)^2(q-2)^2\\
&\quad+8(p-3)-12\,.\,\tfrac{1}{2}(p-3)(p-2)+6\,.\,\tfrac{1}{6}(p-3)(p-2)(2p-5)-\tfrac{1}{4}(p-3)^2(p-2)^2,\\
\Sigma\alpha\alpha' &= 2(p-2)(q-3)-(p-4)\,.\,\tfrac{1}{2}(q-3)(q-2)-\tfrac{1}{6}(q-3)(q-2)(2q-5)\\
&\quad+2(q-2)(p-3)-(q-4)\,.\,\tfrac{1}{2}(p-3)(p-2)-\tfrac{1}{6}(p-3)(p-2)(2p-5),\\
\Sigma\alpha^2\alpha' &= 4(p-2)(q-3)-4(p-3)\,.\,\tfrac{1}{2}(q-3)(q-2)+(p-6)\,.\,\tfrac{1}{6}(q-3)(q-2)(2q-5)+\tfrac{1}{4}(q-3)^2(q-2)^2,\\
&\quad+4(q-2)(p-3)-4(q-3)\,.\,\tfrac{1}{2}(p-3)(p-2)+(q-6)\,.\,\tfrac{1}{6}(p-3)(p-2)(2p-5)+\tfrac{1}{4}(p-3)^2(p-2)^2,\\
\Sigma\mathrm{A} &= \tfrac{1}{2}(p+q-5)(p+q-4),\\
\Sigma\mathrm{A}^2 &= -\Sigma\mathrm{A}\mathrm{A}'=\tfrac{1}{6}(p+q-5)(p+q-4)(2p+2q-9),\\
\Sigma\mathrm{A}^3 &= -\Sigma\mathrm{A}^2\mathrm{A}'=\tfrac{1}{4}(p+q-5)^2(p+q-4)^2,
\end{aligned}$$

which, putting therein $p+q=\alpha$, $pq=\beta$, and from the reduced expressions obtaining the values of $\Sigma\alpha\beta$, &c., give

$$\begin{aligned}
\Sigma\alpha &= \beta-\tfrac{1}{2}\alpha^2+\tfrac{9}{2}\alpha-18,\\
\Sigma\alpha^2 &= \beta(-\alpha+9)+\tfrac{1}{3}\alpha^3-\tfrac{9}{2}\alpha^2+\tfrac{121}{6}\alpha-58,\\
\Sigma\alpha^3 &= \beta^2(-\tfrac{1}{2})+\beta(\alpha^2-\tfrac{27}{2}\alpha+\tfrac{121}{2})-\tfrac{1}{4}\alpha^4+\tfrac{9}{2}\alpha^3-\tfrac{121}{4}\alpha^2+90\alpha-198,\\
\Sigma\alpha\beta &= \beta^2(\tfrac{1}{2})+\beta(-\tfrac{1}{2}\alpha^2+5\alpha-\tfrac{45}{2})+\tfrac{1}{8}\alpha^4-\tfrac{29}{12}\alpha^3+\tfrac{171}{8}\alpha^2-\tfrac{1093}{12}\alpha+191,\\
\Sigma\alpha\beta\gamma &= \beta^3(\tfrac{1}{6})+\beta^2(-\tfrac{1}{4}\alpha^2+\tfrac{11}{4}\alpha-\tfrac{41}{3})+\beta(\tfrac{1}{8}\alpha^4-\tfrac{8}{3}\alpha^3+\tfrac{629}{24}\alpha^2-\tfrac{749}{6}\alpha+\tfrac{1753}{6})\\
&\quad-\tfrac{1}{48}\alpha^6+\tfrac{31}{48}\alpha^5-\tfrac{445}{48}\alpha^4+\tfrac{3617}{48}\alpha^3-\tfrac{8969}{24}\alpha^2+1071\alpha-1560,\\
\Sigma\alpha' &= \beta+\tfrac{1}{2}\alpha^2-\tfrac{15}{2}\alpha+18,\\
\Sigma\alpha\alpha' &= \beta(-\tfrac{1}{2}\alpha)-\tfrac{1}{3}\alpha^3+\tfrac{9}{2}\alpha^2-\tfrac{175}{6}\alpha+58,\\
\Sigma\alpha^2\alpha' &= \beta^2(-\tfrac{1}{6})+\beta(-\tfrac{2}{3}\alpha^2+9\alpha-\tfrac{121}{6})+\tfrac{1}{4}\alpha^4-\tfrac{9}{2}\alpha^3+\tfrac{121}{4}\alpha^2-119\alpha+198,
\end{aligned}$$

$$\begin{aligned}
\Sigma\mathrm{A} &= \tfrac{1}{2}\alpha^2-\tfrac{9}{2}\alpha+10,\\
\Sigma\mathrm{A}^2 &= -\Sigma\mathrm{AA}' = \tfrac{1}{3}\alpha^3-\tfrac{9}{2}\alpha^2+\tfrac{121}{6}\alpha-30,\\
\Sigma\mathrm{A}^3 &= -\Sigma\mathrm{A}^2\mathrm{A}' = \tfrac{1}{4}\alpha^4-\tfrac{9}{2}\alpha^3+\tfrac{121}{4}\alpha^2-90\alpha+100,\\
\Sigma\mathrm{AB} &= \tfrac{1}{8}\alpha^4-\tfrac{29}{12}\alpha^3+\tfrac{139}{8}\alpha^2-\tfrac{661}{12}\alpha+65,\\
\Sigma\mathrm{ABC} &= \tfrac{1}{48}\alpha^6-\tfrac{31}{48}\alpha^5+\tfrac{397}{48}\alpha^4-\tfrac{2689}{48}\alpha^3+\tfrac{5081}{24}\alpha^2-\tfrac{1270}{3}\alpha+\tfrac{1050}{3},
\end{aligned}$$

we then find

$$\begin{aligned}
\Sigma\mathrm{A}+\Sigma\alpha &= \beta-8,\\
\Sigma\mathrm{A}'+\Sigma\alpha' &= \beta-3\alpha+8,\\
\Sigma\mathrm{AB}-\Sigma\alpha\beta &= \beta^2(-\tfrac{1}{2})+\beta(\tfrac{1}{2}\alpha^2-5\alpha+\tfrac{45}{2})-4\alpha^2+36\alpha-126,\\
\Sigma\mathrm{AA}'-\Sigma\alpha\alpha' &= \beta(-\tfrac{1}{2}\alpha)+9\alpha-28,\\
\Sigma\mathrm{ABC}+\Sigma\alpha\beta\gamma &= \beta^3(\tfrac{1}{6})+\beta^2(-\tfrac{1}{4}\alpha^2+\tfrac{11}{4}\alpha-\tfrac{41}{3})+\beta(\tfrac{1}{8}\alpha^4-\tfrac{8}{3}\alpha^3+\tfrac{629}{24}\alpha^2-\tfrac{749}{6}\alpha+\tfrac{1753}{6})\\
&\quad -\alpha^4+\tfrac{58}{3}\alpha^3-162\alpha^2+\tfrac{1943}{3}\alpha-1210,\\
\Sigma\mathrm{A}^2\mathrm{A}'+\Sigma\alpha^2\alpha' &= \beta^2(-\tfrac{1}{6})+\beta(-\tfrac{2}{3}\alpha^2+9\alpha-\tfrac{121}{6})-29\alpha+98;
\end{aligned}$$

and then also

$$\begin{aligned}
&\Sigma\alpha(\Sigma\mathrm{A}+\Sigma\alpha) = \beta^2+\beta(-\tfrac{1}{2}\alpha^2+\tfrac{9}{2}\alpha-26)+4\alpha^2-36\alpha+144,\\
&(\Sigma\mathrm{AB}-\Sigma\alpha\beta)+\Sigma\alpha(\Sigma\mathrm{A}+\Sigma\alpha) = \beta^2(\tfrac{1}{2})+\beta(-\tfrac{1}{2}\alpha-\tfrac{7}{2})+18,\\
&\{(\Sigma\mathrm{AB}-\Sigma\alpha\beta)+\Sigma\alpha(\Sigma\mathrm{A}+\Sigma\alpha)\}(\Sigma\mathrm{A}'+\Sigma\alpha') =\\
&\qquad \beta^3(\tfrac{1}{2})+\beta^2(-2\alpha+\tfrac{1}{2})+\beta(\tfrac{3}{2}\alpha^2+\tfrac{13}{2}\alpha-10)-54\alpha+144,\\
&-(\Sigma\mathrm{A}+\Sigma\alpha)(\Sigma\mathrm{AA}'-\Sigma\alpha\alpha') =\\
&\qquad \beta^2(\tfrac{1}{2}\alpha)+\beta(-13\alpha+28)+72\alpha-224,
\end{aligned}$$

and

$$\begin{aligned}
&\Sigma\mathrm{A}^2\mathrm{A}'+\Sigma\alpha^2\alpha' = (\textit{ut suprà})\\
&\qquad \beta^2(-\tfrac{1}{6})+\beta(-\tfrac{2}{3}\alpha^2+9\alpha-\tfrac{121}{6})-29\alpha+98;
\end{aligned}$$

whence, adding the last three expressions, we find

$$\text{Weight}=\beta^3(\tfrac{1}{2})+\beta^2(-\tfrac{3}{2}\alpha+\tfrac{1}{3})+\beta(\tfrac{5}{6}\alpha^2+\tfrac{5}{2}\alpha-\tfrac{13}{6})-11\alpha+18;$$

and for the order we have

$$\begin{aligned}
(\Sigma\alpha)^2-\Sigma\alpha\beta = {}&\beta^2(\tfrac{1}{2})+\beta(-\tfrac{1}{2}\alpha^2+4\alpha-\tfrac{27}{2})\\
&+\tfrac{1}{8}\alpha^4-\tfrac{25}{12}\alpha^3+\tfrac{135}{8}\alpha^2-\tfrac{851}{12}\alpha+133;
\end{aligned}$$

and then

$$\begin{aligned}
&\Sigma\mathrm{ABC}+\Sigma\alpha\beta\gamma = (\textit{ut suprà})\\
&\qquad \beta^3(\tfrac{1}{6})+\beta^2(-\tfrac{1}{4}\alpha^2+\tfrac{11}{4}\alpha-\tfrac{41}{3})+\beta(\tfrac{1}{8}\alpha^4-\tfrac{8}{3}\alpha^3+\tfrac{629}{24}\alpha^2-\tfrac{749}{6}\alpha+\tfrac{1753}{6})\\
&\qquad -\alpha^4+\tfrac{58}{3}\alpha^3-162\alpha^2+\tfrac{1943}{3}\alpha-1210,\\
&(\Sigma\mathrm{AB}-\Sigma\alpha\beta)\Sigma\alpha =\\
&\qquad \beta^3(-\tfrac{1}{2})+\beta^2(\tfrac{3}{4}\alpha^2-\tfrac{29}{4}\alpha+\tfrac{63}{2})+\beta(-\tfrac{1}{4}\alpha^4+\tfrac{19}{4}\alpha^3-\tfrac{187}{4}\alpha^2+\tfrac{909}{4}\alpha-531)\\
&\qquad +2\alpha^4-36\alpha^3+297\alpha^2-1215\alpha+2268,\\
&\big((\Sigma\alpha)^2-\Sigma\alpha\beta\big)(\Sigma\mathrm{A}+\Sigma\alpha) =\\
&\qquad \beta^3(\tfrac{1}{2})+\beta^2(-\tfrac{1}{2}\alpha^2+4\alpha-\tfrac{35}{2})+\beta(\tfrac{1}{8}\alpha^4-\tfrac{25}{12}\alpha^3+\tfrac{167}{8}\alpha^2-\tfrac{1235}{12}\alpha+241)\\
&\qquad -\alpha^4+\tfrac{50}{3}\alpha^3-135\alpha^2+\tfrac{1702}{3}\alpha-1064;
\end{aligned}$$

whence, adding these three expressions,

$$\text{Order}=\beta^3(\tfrac{1}{6})+\beta^2(-\tfrac{1}{2}\alpha+\tfrac{1}{3})+\beta(\tfrac{1}{3}\alpha^2-\tfrac{1}{2}\alpha+\tfrac{13}{6})-6\,;$$

and by means of the foregoing expression for the weight, we then have

$$\text{Weight}-\text{Order}=\beta^3(\tfrac{1}{3})+\beta^2(-\alpha\qquad)+\beta(\tfrac{1}{2}\alpha^2+3\alpha-\tfrac{13}{3})-11\alpha+24\,;$$

and therefore

$$\begin{aligned}G(m^4)&=\tfrac{1}{4}\beta\times(\text{Weight}-\text{Order})\\&=\tfrac{1}{24}\beta\{2\beta^3+\beta^2(-6\alpha)+\beta(3\alpha^2+18\alpha-26)-66\alpha+144\},\end{aligned}$$

which is right.

Annex No. 4.—*Order of* Torse (m, n) (referred to, Art. 44).

We have to find the order of the developable or Torse generated by a line meeting two curves of the orders m, n respectively; viz. representing by μ, ν the classes of the two curves respectively, it is to be shown that the expression for the Order is

$$\text{Torse } (m, n)=m\nu+n\mu.$$

I remark, in the first place, that, given two surfaces of the orders p and q respectively, the curve of intersection is of the order pq and class $pq(p+q-2)$, or as this may be written, class $=qp(p-1)+pq(q-1)$. Reciprocally for two surfaces of the classes p and q respectively, the Torse enveloped by their common tangent planes is of the class pq and order $qp(p-1)+pq(q-1)$. Now, in the same way that a surface of the order p may degenerate into a Torse of the order p, so a surface of the class p may degenerate into a curve of the class p; and the class of a curve being p, then (disregarding singularities) its order is $=p(p-1)$. Hence replacing p and $p(p-1)$ by μ and m respectively, and in like manner q and $q(q-1)$ by ν and n respectively, we have $m\nu+n\mu$ as the order of the Torse generated by the tangent planes of the curves of the orders m and n respectively; where by tangent plane of a curve is to be understood a plane passing through a tangent line of the curve. The intersection of two consecutive tangent planes is a line meeting the two curves, which line is the generating line of the Torse, and such Torse is therefore the Torse (m, n) in question.

The foregoing investigation is not very satisfactory, but I confirm it by considering the case of two plane curves, orders m and n, and classes μ and ν, respectively. The tangents of the two curves can, it is clear, only meet on the line of intersection of the planes of the curves; and the construction of the Torse is in fact as follows: from any point of the line of intersection draw a tangent to m and a tangent to n, then the line joining the points of contact of these tangents is a generating line of the Torse. The order of the Torse is equal to the number of generating lines which meet an arbitrary line; and taking for the arbitrary line the line of intersection of the two planes, it is easy to see that the only generating lines which meet the line of intersection are those for which one of the points of contact lies on the line of intersection; that is, they are

the generating lines derived from the points in which the line of intersection meets one or other of the two curves; they are therefore in fact the tangents drawn to the curve n from the points in which the line of intersection meets the curve m, together with the tangents drawn to the curve m from the points in which the line of intersection meets the curve n. Now the line meets the curve n in n points, and from each of these there are μ tangents to the curve m; and it meets the curve m in m points, and from each of these there are ν tangents to the curve n; hence the entire number of the tangents in question is $=n\mu+m\nu$, which confirms the theorem.

Annex No. 5.—*Order of* Torse (m^2) (referred to, Art. 46).

We have here to find the order of the developable or Torse generated by a line meeting a curve of the order m twice, viz., the class of the curve being μ, it is to be shown that we have

$$\text{Torse } (m^2)=(m-3)\mu.$$

I deduce the expression from the formula given p. 424 of Dr. SALMON's 'Geometry of Three Dimensions;' viz. putting in his formula $\beta=0$, and μ for his r, we have

$$\text{Order}=m(\mu-4)-\tfrac{1}{2}\alpha=m\mu-(4m+\tfrac{1}{2}\alpha),$$

where (see p. 234 *et seq.*)

$$\mu=m(m-1)-2h,$$
$$\tfrac{1}{2}\alpha=(n-m)=3m(m-2)-6h-m,$$

and thence

$$3\mu-\tfrac{1}{2}\alpha=4m, \text{ or } 4m+\tfrac{1}{2}\alpha=3\mu,$$

so that we have

$$\text{Order }=(m-3)\mu.$$

A more complete discussion of the Torses (m, n) and (m^2) is obviously desirable; but as they are only incidentally connected with the subject of the present memoir, I have contented myself with obtaining the required results in the way which most readily presented itself.